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Article

Controlled Quantum Network Coding Without Loss of Information

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Abstract: Quantum network coding is used to solve the congestion problem in quantum communication, which will promote the transmission efficiency of quantum information and the total throughput of quantum network. We propose a novel controlled quantum network coding without information loss. The effective transmission of quantum states on the butterfly network requires the consent form a third-party controller Charlie. Firstly, two pairs of three-particle non-maximum entangled states are pre-shared between senders and controller. By adding auxiliary particles and local operations, the senders can predict whether a certain quantum state can be successfully transmitted within the butterfly network based on the Z-$\{|0\rangle, |1\rangle\}$ basis. Secondly, when transmission fails upon prediction, the quantum state will not be lost, and it will still be held by the sender. Subsequently, the controller Charlie re-prepares another three-particle non-maximum entangled state to start a new round. When the predicted transmission is successful, the quantum state can be transmitted successfully within the butterfly network. If the receiver wants to receive the effective quantum state, the quantum measurements from Charlie are needed. Thirdly, when the transmission fails, Charlie does not need to integrate the X-$\{|+\rangle, |-\rangle\}$ basis to measure its own particles, by which quantum resources are saved. Charlie not only controls the effective transmission of quantum states, but also the usage of classical and quantum channels. Finally, the implementation of the quantum circuits, as well as a flow chart and safety analysis of our scheme, is proposed.

Keywords: Controlled quantum network coding; without information loss; quantum teleportation; perfect transmission

1 Introduction

In 2000, classic network coding was first proposed by Ahlswede et al. [1], which improved the transmission efficiency of classic information by coding at bottleneck nodes in the network. In 2007, Hayashi et al. [2] considered the features and advantages of classical network coding, and proposed the idea of quantum network coding for the first time. However, because the exact replication of an unknown quantum state is impossible in quantum mechanics [3,4], only approximate transmissions between quantum states can be realized on the butterfly network without auxiliary entanglements. The scheme has been applied to solve congestion problems in quantum information transmissions by unitary operations on the bottleneck nodes, and has been proved to improve the transmission efficiency of quantum information. Since quantum approximation clone machines were used [4], the fidelity of the quantum states received by the receiver could not reach 1. Based on Hayashi's scheme, a controllable quantum network coding based on a single controller was proposed by Shang et al. [5], which realized decoding control in the receiver in a conventional quantum network coding. However, because Shang's scheme employed quantum approximation cloning, the fidelity of the received quantum states still could not reach 1. Later, Kobayashi et al. proved that quantum network coding with a fidelity of 1 can be achieved with assistance from auxiliary resources [6–8]. Since then, perfect and cross transmission of quantum states on quantum networks has become a research interest for many researchers. By definition, perfect transmission means that the fidelity of the quantum states received by the receiver is 1.

In 2007, Hayashi [9] realized perfect and cross transmission of quantum states by pre-sharing two pairs of entangled states among senders on the butterfly network. By adding auxiliary resources and combining with classical network coding, quantum states were transmitted in the scheme with a fidelity of 1. In 2012, another quantum network coding scheme based on quantum repeaters [10–12] was proposed by Satoh et al. [13]. In this scheme, each node in the butterfly network was regarded as a quantum repeater, and every two adjacent quantum repeaters shared an EPR pair. With local operations and classical communications, entanglement between the receiver and the sender was created. After that, quantum teleportation [14,15] was applied to realize perfect and cross transmission of quantum states within the butterfly network. Schemes [9,13] used the maximally-entangled states as the auxiliary resources to realize perfect and cross transmission of quantum states on the butterfly network. However, it is difficult to prepare such states in practice, and non-maximum entangled states are more feasible, which was employed by Ma et al. [16] to develop a probabilistic quantum network coding. Moreover, they have been applied by Shang et al. [17] to propose another quantum network coding based on universal quantum repeater networks.

In addition, Satoh et al. [18] continued to use entanglement swap and graph states [19] to achieve perfect and cross transmission of quantum states. Subsequently, Li et al. [20] extended the conclusion from [13] to quantum multi-unicast networks, which solved the quantum 3-pair communications problem. Besides, Li et al. also proposed a solution to the problem of quantum k-pair communications in 2018. At present, research on quantum network coding has become a hot spot with more and more schemes being proposed [21–27].

When non-maximum entangled states are used as a quantum channel, the quantum states will be transmitted with a certain probability. If transmission fails, the quantum states will be lost. Therefore, the preservation of quantum states during transmission has become an urgent problem. In 2015, Roa et al. [28] proposed probabilistic quantum teleportation without information loss, in which the non-maximum entangled states are pre-shared between the sender and the receiver. By adding auxiliary particles and local operations [29], the transmission of quantum states can be realized without information loss, and the quantum states will remain at the sender if transmission fails. As long as the entangled resources are sufficient, the transmission of quantum states can be tried repeatedly until success. Such idea is adopted into this work, in which the advantages of classical network coding are combined to create a controlled quantum network coding scheme that could achieve perfect and cross transmission of quantum states without information loss.

Since coupling between the quantum states and the surrounding environment is inevitable in practice [30], it is of more practical significance to use non-maximum entangled states as the auxiliary resources to achieve perfect transmission of quantum states [31]. However, under such circumstances, the successful transmission of quantum states on the butterfly network is not guaranteed [16,17]. If the transmission fails, the quantum states will be lost, resulting in invalid communication and waste of channel resources. Here in this paper, we consider pre-sharing two pairs of three-particle non-maximum entangled states between the senders and the controller Charlie on the butterfly network. Our scheme combines quantum teleportation with classical network coding to solve the bottleneck problem of quantum state transmission. Under Charlie's control, perfect and cross transmission of the quantum states can be achieved. Particularly, the senders can predict whether the quantum states can be successfully transmitted over the butterfly network with the help of auxiliary particles. When transmission fails, the quantum states will not be lost, and they will remain at the sender to be used for the next transmission. Moreover, both classical and quantum channels are not occupied if transmission fails. In this scheme, Charlie controls not only whether the receiver can receive the quantum states, but also the usage of classical and quantum channels over the butterfly network. Therefore, our scheme improves the utilization efficiency of both channels.

In the following sections, the paper content is organized as below. Some preliminary definitions and equations involved in our scheme will be given in Section 2. In Section 3, the implementation procedure of our controlled quantum network coding without information loss will be discussed in detail. In addition, the implementation of the quantum circuit implementation, as well as the flow chart and safety analysis for our scheme will be demonstrated in this section as well, which could be of great reference value for future researches. Finally, our conclusions will be stated in Section 4.

2 Preliminaries

2.1 Three-Particle Non-Maximum Entangled State

In our scheme, we will use a three-particle non-maximum entangled state as quantum channel.

$$|\Phi\rangle_{ABC} = \alpha\ |000\rangle_{ABC} + \beta\ |111\rangle_{ABC} \tag{1}$$

where α, β are positive real numbers and $\alpha \le \beta$. It satisfies the normalization condition $\alpha^2 + \beta^2 = 1$, and particles A, B and C belong to different parties.

2.2 Local Operations

Some single-particle gate operations and two-particle local operations [29] are applied. The single-particle gate operations are:

$$\sigma_{\mathrm{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma_{\mathrm{z}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{2}$$

The influences from the single-particle gates on the quantum states are:

$$\sigma_x |0\rangle = |1\rangle, \sigma_x |1\rangle = |0\rangle, \sigma_z |0\rangle = |0\rangle, \sigma_z |1\rangle = -|1\rangle, H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{3}$$

The two-particle local operations are:

$$C_{ij} = |0\rangle_i \langle 0|_i \otimes I_j + |1\rangle_i \langle 1|_i \otimes \sigma_x^{(j)} \tag{4}$$

This operation is called a controlled NOT gate, in which particle i is a control qubit and particle j is a target qubit.

In our scheme, a controlled unitary operation is applied to ensure that quantum states are not lost.

$$C_{ij}^{U_j} = |0\rangle_i \langle 0|_i \otimes I_j + |1\rangle \langle 1| \otimes U_j \tag{5}$$

Here,

$$U_j = \begin{bmatrix} \frac{\alpha}{\beta} & \sqrt{1 - \frac{\alpha^2}{\beta^2}} \\ \sqrt{1 - \frac{\alpha^2}{\beta^2}} & -\frac{\alpha}{\beta} \end{bmatrix} \tag{6}$$

2.3 Controlled Quantum Teleportation

In our scheme, controlled quantum teleportation [32–34] is introduced into quantum network coding. Its realization can be described as follows.

Alice, Bob and the controller Charlie share a three-particle non-maximum entangled state $|\Phi\rangle_{ABC}$. Particle A belongs to the sender Alice, particle B belongs to the receiver Bob, and particle C belongs to the controller Charlie. Now Alice wants to transmit an unknown quantum state $|\psi\rangle_a$ to Bob. The combined state of $|\Phi\rangle_{ABC}$ and $|\psi\rangle_a$ is:

$$\begin{aligned} |\psi\rangle_a |\Phi\rangle_{ABC} = \frac{1}{2} \big[& |\Psi^+\rangle_{aA} (a|00\rangle + b|11\rangle)_{BC} + |\Psi^-\rangle_{aA} (a|00\rangle - b|11\rangle)_{BC} \\ & + |\Phi^+\rangle_{aA} (b|00\rangle + a|11\rangle)_{BC} + |\Phi^-\rangle_{aA} (b|00\rangle - a|11\rangle)_{BC} \big] \end{aligned} \tag{7}$$

where $|\Psi^\pm\rangle = \alpha|00\rangle \pm \beta|11\rangle$, $|\Phi^\pm\rangle = \alpha|10\rangle \pm \beta|01\rangle$. We use a two-particle basis $\{|\Psi^\pm\rangle, |\Phi^\pm\rangle\}$ to measure particles aA. The following equations illustrate the four states measured by $\{|\Psi^\pm\rangle, |\Phi^\pm\rangle\}$ with an equal probability.

$$\begin{aligned} |\Upsilon_1\rangle_{BC} &= a|00\rangle_{BC} + b|11\rangle_{BC} \\ |\Upsilon_2\rangle_{BC} &= a|00\rangle_{BC} - b|11\rangle_{BC} \\ |\Upsilon_3\rangle_{BC} &= b|00\rangle_{BC} + a|11\rangle_{BC} \\ |\Upsilon_4\rangle_{BC} &= b|00\rangle_{BC} - a|11\rangle_{BC} \end{aligned} \tag{8}$$

After that, Charlie integrates the X-$\{|+\rangle, |-\rangle\}$ basis to measure particle C. The Charlie needs to tell Bob the measurement results, so that Bob can recover the unknown quantum state transmitted by Alice. For example, if the measurement result from Alice is $|\Psi^+\rangle_{aA}$, the quantum state $|\Upsilon_1\rangle_{BC}$ is subsequently obtained, and Charlie measures particle C. When the measurement result from Charlie is $|+\rangle_C$, Bob executes an identity operator I to particle B based on $|\Psi^+\rangle_{aA}$

and $|+\rangle_C$ to obtain the quantum state $|\psi\rangle_B$. When the measurement result from Charlie is $|-\rangle_C$, Bob executes an Pauli operator σ_z to particle B based on $|\Psi^+\rangle_{aA}$ and $|-\rangle_C$ to obtain the quantum state $|\psi\rangle_B$.

Therefore, in order to receive the unknown quantum state transmitted by Alice, measurement results from Alice and Charlie are both needed for Bob to realize satisfactory quantum state recovery.

3 Our Work

In this section, we propose a controlled quantum network coding scheme without information loss. Our scheme will be discussed based on the measurement results from auxiliary particles. In addition, the flow chart and safety analysis of our scheme will also be given.

3.1 Controlled Quantum Network Coding without Loss of Information

In our scheme, a third-party controller Charlie is added. As is shown in Fig. 1, the capacity of the bidirectional classical channel is 1 bit, and the dotted line represents a quantum channel with a capacity of 1 qubit, while the solid line stands for a classical channel with a capacity of 2 bit. Effective transmission of quantum states between the sender and the receiver require consent from Charlie C, so that the receivers can receive quantum states as they originally are. In our scheme, Charlie not only controls the transmission of quantum states, but also inhibits the unnecessary transmission of classical and quantum information on the butterfly network. When transmission fails, the transmitted quantum states will not be lost and still at the sender. The specific protocol is demonstrated as follows:

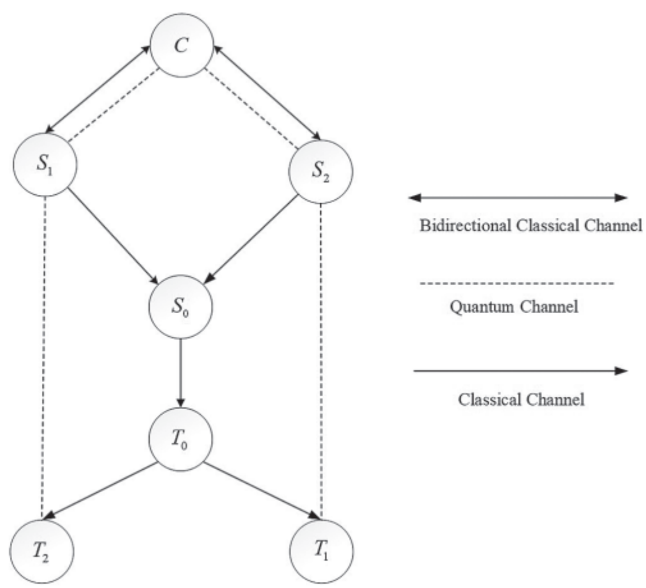

Figure 1: Quantum butterfly network based on controller Charlie

In our scheme, the three-particle non-maximum entangled states, which are prepared by Charlie, are pre-shared between the senders and Charlie on the butterfly network. Two pairs of three-particle non-maximum entangled states, namely $|\Phi_1\rangle = (\alpha\, |000\rangle + \beta\, |111\rangle)_{s_{1,1}s_{2,1}c_1}$ and $|\Phi_2\rangle =$

$(\gamma\ |000\rangle + \delta\ |111\rangle)_{s_{1,2}s_{2,2}c_2}$, are necessary for transmission of quantum states. After preparation of the entangled states, Charlie sends them to the senders of S_1 and S_2 through the quantum channels of $Q(C,S_1)$ and $Q(C,S_2)$, respectively. The particles of $s_{1,1}$, $s_{1,2}$ are owned by S_1, the particles of $s_{2,1}$, $s_{2,2}$ are owned by S_2, and the particles of c_1,c_2 are owned by Charlie. Both S_1 and S_2 prepare arbitrary quantum states to be transmitted, which are $|\psi_1\rangle_{s_1} = (a_1\ |0\rangle + b_1\ |1\rangle)_{s_1}$ and $|\psi_2\rangle_{s_2} = (a_2\ |0\rangle + b_2\ |1\rangle)_{s_2}$, respectively. Specifically, a_1, a_2, b_1 and b_2 are complex numbers and satisfy the normalization condition $|a_1|^2 + |b_1|^2 = 1$, $|a_2|^2 + |b_2|^2 = 1$. Our scheme contains four stages, namely local operations, encoding, transmission and decoding.

Firstly, in local operations, the combined state of the unknown state $|\psi_i\rangle_{s_i}$ and the three-particle non-maximum entangled state $|\Phi_{\mathrm{i}}\rangle$ is expressed as

$$\begin{aligned}|\Pi\rangle &= |\psi_i\rangle_{s_i}\ |\Phi_{\mathrm{i}}\rangle \\ &= (a_i\ |0\rangle + b_i\ |1\rangle)_{s_i}\ (\alpha_i\ |000\rangle + \beta_i\ |111\rangle)_{s_{i,i}s_{i\oplus 1,i}c_i}\end{aligned} \tag{9}$$

$i \in \{1, 2\}$ in our entire protocol.

Sender S_i applies $C_{s_{i,i}s_i}$ to its own bipartite system $s_{i,i}s_i$, and the initial state $|\Pi\rangle$ becomes

$$\begin{aligned}|\Pi_1\rangle &= \mathrm{C}_{s_{i,i}s_i}\ |\psi_i\rangle_{s_i}\ |\Phi_{\mathrm{i}}\rangle \\ &= (\alpha_i a_i\ |0000\rangle + \alpha_i b_i\ |1000\rangle + \beta_i a_i\ |1111\rangle + \beta_i b_i\ |0111\rangle)_{s_i s_{i,i} s_{i\oplus 1,i} c_i}\end{aligned} \tag{10}$$

Sender S_i adds an auxiliary particle e_i, which is initialized to $|0\rangle_{e_i}$. Subsequently, S_i applies $C_{s_ie_i}$ on the bipartite system s_ie_i, and the quantum state $|\Pi_1\rangle$ becomes

$$\begin{aligned}|\Pi_2\rangle &= \mathrm{C}_{s_ie_i}\mathrm{C}_{s_{i,i}s_i}\ |\psi_i\rangle_{s_i}\ |\Phi_{\mathrm{i}}\rangle\ |0\rangle_{e_i} \\ &= (\alpha_i a_i\ |00000\rangle + \alpha_i b_i\ |10001\rangle + \beta_i a_i\ |11111\rangle + \beta_i b_i\ |01110\rangle)_{s_i s_{i,i} s_{i\oplus 1,i} c_i e_i}\end{aligned} \tag{11}$$

After obtaining $|\Pi_2\rangle$, S_i applies $C_{s_{i,i}s_i}^{U_{s_i}}$ on the bipartite system $s_{i,i}s_i$, and the quantum state $|\Pi_2\rangle$ becomes

$$\begin{aligned}|\Pi_3\rangle &= \mathrm{C}_{s_{i,i}s_i}^{U_{s_i}}\mathrm{C}_{s_ie_i}\mathrm{C}_{s_{i,i}s_i}\ |\psi_i\rangle_{s_i}\ |\Phi_{\mathrm{i}}\rangle\ |0\rangle_{e_i} \\ &= (\alpha_i a_i\ |00000\rangle + \alpha_i b_i\ |10001\rangle - \beta_i a_i\ |11111\rangle + \beta_i b_i\ |01110\rangle)_{s_i s_{i,i} s_{i\oplus 1,i} c_i e_i} \\ &\quad + \sqrt{\beta_i^2 - \alpha_i^2}\ (a_i\ |01111\rangle + b_i\ |11110\rangle)_{s_i s_{i,i} s_{i\oplus 1,i} c_i e_i}\end{aligned} \tag{12}$$

After that, S_i applies $C_{s_ie_i}$ on the bipartite system s_ie_i, and the quantum state $|\Pi_3\rangle$ becomes

$$\begin{aligned}|\Pi_4\rangle &= \mathrm{C}_{s_ie_i}\mathrm{C}_{s_{i,i}s_i}^{U_{s_i}}\mathrm{C}_{s_ie_i}\mathrm{C}_{s_{i,i}s_i}\ |\psi_i\rangle_{s_i}\ |\Phi_{\mathrm{i}}\rangle\ |0\rangle_{e_i} \\ &= (\alpha_i a_i\ |0000\rangle + \alpha_i b_i\ |1000\rangle - \beta_i a_i\ |1111\rangle + \beta_i b_i\ |0111\rangle)_{s_i s_{i,i} s_{i\oplus 1,i} c_i}\ |0\rangle_{e_i} \\ &\quad + \sqrt{\beta_i^2 - \alpha_i^2}\ (a_i\ |0\rangle + b_i\ |1\rangle)_{s_i}\ |111\rangle_{s_{i,i} s_{i\oplus 1,i} c_i}\ |1\rangle_{e_i}\end{aligned} \tag{13}$$

Secondly, in the stage of encoding, S_i uses the Z-$\{|0\rangle, |1\rangle\}$ basis to measure the auxiliary particle e_i. When the measurement result is $|0\rangle$, it suggest that perfect and cross transmission of quantum states on the butterfly network is possible, and the following quantum state can be obtained.

$$|\mathrm{T}_{00}\rangle = (\alpha_i a_i\ |0000\rangle + \alpha_i b_i\ |1000\rangle - \beta_i a_i\ |1111\rangle + \beta_i b_i\ |0111\rangle)_{s_i s_{i,i} s_{i\oplus 1,i} c_i} \tag{14}$$

Subsequently, S_i transmits the measurement result to Charlie through the classical channel $C(S_i, C)$. When Charlie receives a transmission request from the senders, it will employ the

X-$\{|+\rangle, |-\rangle\}$ basis to measure particle c_i, and the measurement results are encoded according to Tab. 1.

When the measurement result is $|+\rangle_{c_i}$, the quantum state collapses to $|\mathrm{K}_1\rangle$.

$$|\mathrm{K}_1\rangle = (\alpha_i a_i |000\rangle + \alpha_i b_i |100\rangle - \beta_i a_i |111\rangle + \beta_i b_i |011\rangle)_{s_i s_{i,i} s_{i\oplus 1,i}} \tag{15}$$

When the measurement result is $|+\rangle_{c_i}$, the quantum state collapses to $|\mathrm{K}_2\rangle$.

$$|\mathrm{K}_2\rangle = (\alpha_i a_i |000\rangle + \alpha_i b_i |100\rangle + \beta_i a_i |111\rangle - \beta_i b_i |011\rangle)_{s_i s_{i,i} s_{i\oplus 1,i}} \tag{16}$$

Particularly, Tab. 1 only needs to be held by Charlie and S_i, and the receiver T_i does not need to know. After encoding the measurement results, Charlie sends the classic bit Y_i to its corresponding sender $S_{i\oplus 1}$ through the classical channel $C(C, S_{i\oplus 1})$. When $S_{i\oplus 1}$ receives Y_i, a unitary operation $U(Y_i)$ is applied to the particle $s_{i\oplus 1,i}$ according to Tab. 1. Next, S_i employs the Z-$\{|0\rangle, |1\rangle\}$ basis and the X-$\{|+\rangle, |-\rangle\}$ basis to measure particles s_i and $s_{i,i}$, respectively, and the measurement results are encoded according to Tab. 2.

Table 1: Controller-Charlie: coding and operation

Measurement result	Classic bit (Y_i)	Unitary operation $U(Y_i)$
$\|+\rangle_{c_i}$	0	I
$\|-\rangle_{c_i}$	1	σ_z

Table 2: Sender-Receiver: coding & decoding

Measurement result	Classic bit (X_i)	$U(X_i)\left(U(X_i)^{-1}\right)$
$\|0\rangle_{s_i} \|+\rangle_{s_{i,i}}$	00	I
$\|1\rangle_{s_i} \|-\rangle_{s_{i,i}}$	01	σ_X
$\|0\rangle_{s_i} \|-\rangle_{s_{i,i}}$	10	σ_Z
$\|1\rangle_{s_i} \|+\rangle_{s_{i,i}}$	11	$\sigma_\mathrm{X}\sigma_\mathrm{Z}$

With the help of Tab. 2, S_i encodes its measurement results into a classic bit X_i. After measurements with the single-particle bases, S_i obtains the measured quantum state $U(X_{i\oplus 1})|\psi\rangle_{s_{i,i\oplus 1}}$, to which S_i applies the unitary operation $U(X_i)$ to find $U(X_1 \oplus X_2)|\psi_{i\oplus 1}\rangle_{s_{i,i\oplus 1}}$. Specifically, since in a quantum system, $U(X_i)U(X_{i\oplus 1})|\psi_{i\oplus 1}\rangle_{s_{i,i\oplus 1}} = |\pm 1| U(X_1 \oplus X_2)|\psi_{i\oplus 1}\rangle_{s_{i,i\oplus 1}}$, the global phase can be ignored.

Thirdly, in the transmission stage, S_i sends the quantum state $U(X_1 \oplus X_2)|\psi_{i\oplus 1}\rangle_{s_{i,i\oplus 1}}$ to the receiver $T_{i\oplus 1}$ via the quantum channel $Q(S_i, T_{i\oplus 1})$, and the classic bit X_i to the intermediate node S_0 via the classical channel $C(S_i, S_0)$. After successful transmission of X_i to S_0, an EX-OR (Exclusive-OR) operation is performed to obtain $X_1 \oplus X_2$, which is then transmitted to another node T_0 via the classic channel $C(S_0, T_0)$. At T_0, $X_1 \oplus X_2$ is copied and transmitted to T_i via the classic channel $C(T_0, T_i)$.

Finally, in the decoding stage, T_i applies the unitary operation $U(X_1 \oplus X_2)^{-1}$ to $U(X_1 \oplus X_2)|\psi_i\rangle_{s_{i\oplus 1,i}}$ based on $X_1 \oplus X_2$; that is, $U(X_1 \oplus X_2)^{-1} U(X_1 \oplus X_2)|\psi_i\rangle_{s_{i\oplus 1,i}} = |\psi_i\rangle_{s_{i\oplus 1,i}}$. By the end of the unitary operation, perfect and cross transmission of the quantum states can be realized with the help of the controller Charlie on the butterfly network.

Next, we present an implementation of our scheme on a quantum circuit. As is shown in Fig. 2, Charlie prepares and distributes entangled particles to the senders of S_1 and S_2. In Fig. 2, single lines represent quantum channels, and double lines stand for classical channels. In this implementation, controlled quantum network coding without information loss is realized with the help of both classical and quantum channels.

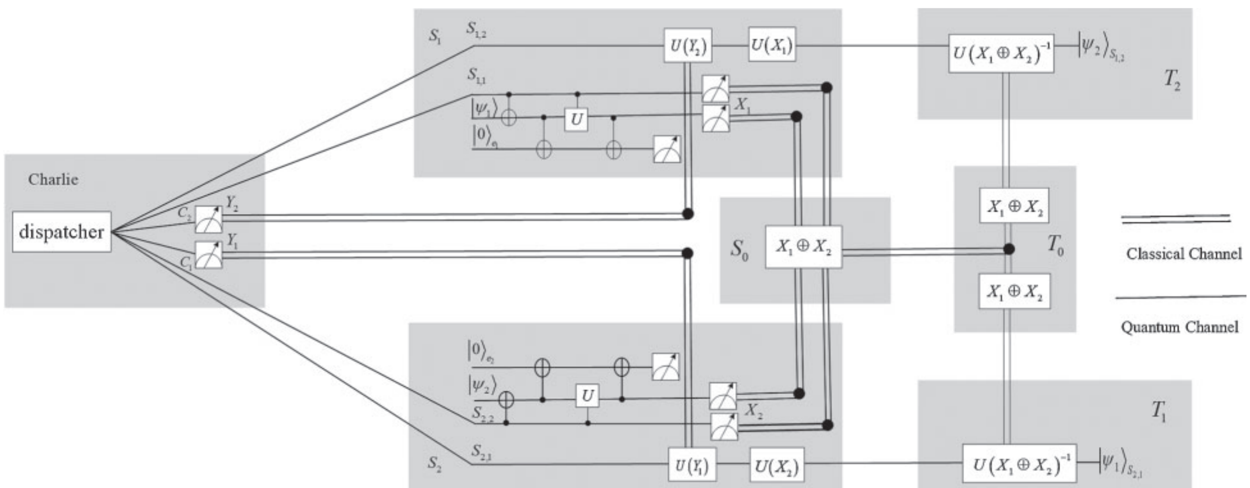

Figure 2: Quantum circuit implementation

Specifically, only senders are controlled by the controller Charlie, which is a feature of this scheme. The receiver only needs to perform unitary operations on the quantum states it received according to Tab. 2, and storage of Tab. 1 becomes unnecessary for the receiver. If S_i sends the quantum states to T_i without the consent from Charlie, then T_i will not be able to effectively recover the original quantum states.

3.2 Discussions

In our scheme, the measurement results of the auxiliary particle e_i given by S_i are sent to Charlie via the classical channel $C(S_i, C)$. When the measurement results of both auxiliary particles are $|0\rangle$, Charlie would control the transmission of quantum states on the butterfly network to be successful. On the other hand, if both measurement results are $|1\rangle$, the quantum states will not be successfully transmitted, as described in the following:

$$|\mathrm{T}_{11}\rangle = (a_i|0\rangle + b_i|1\rangle)_{s_i}|111\rangle_{s_{i,i}s_{i\oplus 1,i}c_i} \tag{17}$$

However, the quantum states will not be lost. S_i will get a quantum state $(a_i|0\rangle + b_i|1\rangle)_{s_i}$ with a probability of $\beta_i^2 - \alpha_i^2$. Besides, Charlie does not need to employ the X-$\{|+\rangle, |-\rangle\}$ basis to measure its own particles. In this way, Charlie only needs to re-prepare two pairs of three-particle non-maximum entangled states for a new cycle until the measurements given by the senders for both auxiliary particles are $|0\rangle$.

When the measurement given by one party on its own auxiliary particle is $|0\rangle$, and that from the other party is $|1\rangle$, only one quantum state can be transmitted successfully on the butterfly network. We assume that the measurement given by S_1 on e_1 is $|0\rangle_{e_1}$ and that given by S_2 is $|1\rangle_{e_1}$. At such circumstances, the quantum state will collapse to $|\mathrm{T}_{01}\rangle$.

$$|\mathrm{T}_{01}\rangle = (\alpha_i a_i\,|0000\rangle + \alpha_i b_i\,|1000\rangle - \beta_i a_i\,|1111\rangle + \beta_i b_i\,|0111\rangle)_{s_1 s_{1,1} s_{2,1} c_1} \otimes (a_i\,|0\rangle + b_i\,|1\rangle)_{s_2}\,|111\rangle_{s_{2,2} s_{1,2} c_2} \tag{18}$$

Here, Charlie re-prepares one three-particle non-maximum entangled state, and distributes the particles to the senders for retransmission of the quantum state. The party who failed in the beginning joins a new round of our scheme until Charlie receives the measurements of its auxiliary particles given by both senders are $|0\rangle$. Subsequently, the Z-$\{|0\rangle, |1\rangle\}$ basis is employed to measure the remaining particles. In this way, when transmission fails, the quantum and classical channels will not be occupied. Additionally, a buffer time T is set in our scheme. If the measurements of the auxiliary particles remain $|1\rangle$ within T for one party, a new round will start.

3.3 Scheme Flow Chart

In order to demonstrate our scheme more clearly, we hereby give a flow chart in Fig. 3. By measuring the auxiliary particles, senders can predict whether the quantum states can be transmitted on the butterfly network in a controlled way. Only the parties with a measurement result of $|1\rangle$, instead of the quantum states, can inform the controller Charlie to re-prepare three-particle non-maximum entangled states.

3.4 Safety Analysis

Quantum network coding is used to solve the congestion problem in the transmission of quantum information, as well as to improve the transmission efficiency, increase network throughput and promote network security. In our scheme, if the sender wants to send a quantum state to the receiver, it needs the consent from a third party Charlie for effective transmission. Therefore, with our scheme, an eavesdropper Eve shall not obtain the original quantum information. In addition, we have not considered inevitable information destruction.

In our scheme, it is Charlie's responsibility to prepare the three-particle non-maximum entangled states and distribute the auxiliary particles to the senders. Information security in this procedure is guaranteed by the BB84 protocol [35]. When the measurement results given by both senders on the auxiliary particles are $|0\rangle$, it shows that the quantum states can be transmitted over the butterfly network under the control of Charlie. In our scheme, Charlie performs measurement according to the X-$\{|+\rangle, |-\rangle\}$ basis, and the senders acts upon both the Z-$\{|0\rangle, |1\rangle\}$ basis and the X-$\{|+\rangle, |-\rangle\}$ basis. After encoding, a quantum state $U(X_1 \oplus X_2)\,|\psi\rangle$ and its corresponding classical information are obtained at the sender. If Eve gets $U(X_1 \oplus X_2)\,|\psi\rangle$ via the quantum channel but fails to acquire the classical information $X_1 \oplus X_2$, the quantum state $|\psi\rangle$ will not be obtained. On the other hand, if Eve obtains the classical information X_1, X_2 and $X_1 \oplus X_2$ from the classical channel without $U(X_1 \oplus X_2)\,|\psi\rangle$, information in $|\psi\rangle$will still be secure. Moreover, even if Eve gets $U(X_1 \oplus X_2)\,|\psi\rangle$ and $X_1 \oplus X_2$ simultaneously, $|\psi\rangle$ will still not be decoded without the coding table, which has been well communicated between the sender and the receiver before transmission.

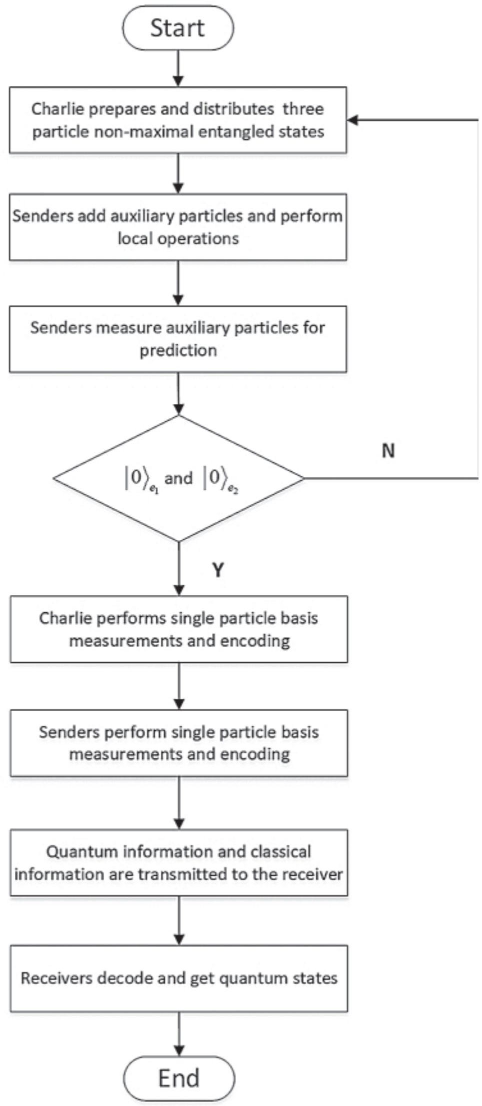

Figure 3: The flow chart of our scheme

To summarize the analysis above, as long as the coding table, which is not seen during the transmission, is not leaked, our scheme is secure. Therefore, our scheme ensures sufficient information security against external eavesdroppers.

4 Conclusions

In the paper, we propose a controlled quantum network coding without information loss by the employment of three-particle non-maximum entangled states on the butterfly network. In our scheme, a third party Charlie is necessary as the controller for perfect and cross transmission of quantum states.

Compared with previous schemes, our scheme is advantageous in several aspects. First of all, compared with the scheme in [5], our scheme realizes perfect and cross transmission of quantum states. Secondly, compared with the scheme in [9], non-maximum entangled states are employed to realize controlled quantum network coding without information loss instead of maximum ones. Specifically, our scheme avoids preparation of the Bell basis and employs single particle bases to measure particles, which is easier for practical applications. Thirdly, compared with the scheme in [16], we consider the probability of failed transmission with the non-maximum entangled states employed as the quantum channel. When the auxiliary particles are measured to be $|1\rangle$, we avoid the re-preparation of quantum states and invalid information transmission on the butterfly network, which improve the utilization efficiency of the channels. Finally, we give an implementation of our scheme on the quantum circuit, which is of great reference value for future studies.

As for the future prospects, we hope that our scheme can be applied in practice. Moreover, this scheme could be extended to a quantum k-pair butterfly network to achieve perfect, cross and controlled transmission of k quantum states with further researches. We also hope that our work can contribute to the development of quantum communication [36–41].

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