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ARTICLE

On Degenerate Array Type Polynomials

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Dedicated to retired Professor Ji-Shan Tian, former vice president of Henan University, China

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ABSTRACT

In the paper, with the help of the Faá di Bruno formula and an identity of the Bell polynomials of the second kind, the authors define degenerate λ-array type polynomials, establish two explicit formulas, and present several recurrence relations of degenerate λ-array type polynomials and numbers.

KEYWORDS

Degenerate array polynomial; Stirling number of the second kind; generating function; explicit formula; recurrence relation

1 Introduction

In this paper, we use the following notation:

$$\mathbb{Z}=\{0,\pm 1,\pm 2,\dots\},\quad \mathbb{N}=\{1,2,\dots\},\quad \mathbb{N}_0=\{0,1,2,\dots\},\quad \mathbb{N}_-=\{-1,-2,\dots\}.$$

The Stirling numbers of the second kind $S(n,m)$ for $n\ge m\ge 0$ can be generated by

$$\frac{(e^t-1)^m}{m!}=\sum_{n=m}^{\infty}S(n,m)\frac{t^n}{n!} \tag{1}$$

and can be computed as

$$S(n,m)=\frac{1}{m!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}(m-i)^n.$$

See [[1], p.206] and the paper [2].

The λ-array type polynomials $S(n,m;x;\lambda)$ were defined in [3] by the generating function

$$\frac{(\lambda e^t-1)^m}{m!}e^{xt}=\sum_{n=0}^{\infty}S(n,m;x;\lambda)\frac{t^n}{n!}. \tag{2}$$

See also the papers [4,5]. It is clear that $S(n,m;0;1)=S(n,m)$. In the paper [6], Simsek obtained and constructed several generating functions and many relations of generalized Stirling type numbers, the array type polynomials, and Eulerian type polynomials. In the paper [7], Bayad et al. deduced interesting and meaningful identities associated with λ-array type polynomials, λ-Stirling numbers of the second kind, and the Apostol–Bernoulli numbers, while they dealt with λ-array polynomials by applying λ-delta operator. Readers interested to the Apostol-Bernoulli numbers and polynomials may consult to the papers [8–10] and closely related references therein.

In the paper [11], Carlitz introduced degenerate Bernoulli and Euler polynomials $B_n(x;\gamma)$ and $E_n(x;\gamma)$ by

$$\frac{t}{(1+\gamma t)^{1/\gamma}-1}(1+\gamma t)^{x/\gamma}=\sum_{n=0}^{\infty}B_n(x;\gamma)\frac{t^n}{n!} \tag{3}$$

and

$$\frac{2}{(1+\gamma t)^{1/\gamma}+1}(1+\gamma t)^{x/\gamma}=\sum_{n=0}^{\infty}E_n(x;\gamma)\frac{t^n}{n!} \tag{4}$$

respectively. For $x=0$, the quantities $B_n(0;\gamma)$ and $E_n(0;\gamma)$ are called as degenerate Bernoulli and Euler numbers. Since $\lim_{\gamma\to 0}(1+\gamma t)^{1/\gamma}=e^t$, Eqs. (3) and (4) reduce to the generating functions for classical Bernoulli and Euler polynomials, respectively.

We now define degenerate λ-array type polynomials $S(n,m;x;\lambda;\gamma)$ by

$$\frac{[\lambda(1+\gamma t)^{1/\gamma}-1]^m}{m!}(1+\gamma t)^{x/\gamma}=\sum_{n=0}^{\infty}S(n,m;x;\lambda;\gamma)\frac{t^n}{n!}. \tag{5}$$

It is easy to see that

$$\lim_{\gamma\to 0}S(n,m;x;\lambda;\gamma)=S(n,m;x;\lambda) \tag{6}$$

which is defined by (2). When $x=0$, we call the quantities $S(n,m;0;\lambda;\gamma)$ degenerate λ-array type numbers.

In this paper, utilizing the Faá di Bruno formula and an identity of the Bell polynomials of the second kind, we establish several explicit formulas and recurrence relations of (degenerate) λ-array type numbers and polynomials.

Let us notice that the Faá di Bruno formula, which can be viewed as an extension of chain rule to higher derivatives, has been applied to establish explicit and closed-form formulas of many important numbers and polynomials in analytic and combinatorial number theory. For more details, please refer to, for example, the papers [12–18] and closely related references therein.

2 Some Identities of the Bell Polynomials of the Second Kind

The Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ can be defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n-k+1 \\ \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n-k+1} i\ell_i = n \\ \sum_{i=1}^{n-k+1} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

See [[1], p. 134]. For $n \in \mathbb{N}$, the Faà di Bruno formula is described in [[1], p.139] in terms of the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=1}^{n} f^{(k)}(h(t)) B_{n,k}\big(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\big). \tag{7}$$

The formula

$$B_{n,k}\left(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\right) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1}(\ell - q\lambda) \tag{8}$$

has been applied and reviewed in [[16], Lemma 2.2], [[17], Remark 6.1], and [[19], Section 1.3]. The explicit formula (8) is equivalent to

$$B_{n,k}(\langle\lambda\rangle_1, \langle\lambda\rangle_2, \dots, \langle\lambda\rangle_{n-k+1}) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \langle\lambda\ell\rangle_n, \tag{9}$$

which was presented in [[20], Theorems 2.1 and 4.1], where the falling factorial $\langle x\rangle_n$ is defined for $x \in \mathbb{C}$ by

$$\langle x\rangle_n = \prod_{k=0}^{n-1}(x-k) = \begin{cases} x(x-1)\dots m(x-n+1), & n \in \mathbb{N}; \\ 1, & n = 0. \end{cases}$$

When $n \in \mathbb{N}$, the explicit formulas (8) and (9) can be rearranged as

$$B_{n,k}\left(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\right) = \begin{cases} (-1)^k \dfrac{\lambda^n n!}{k!} \displaystyle\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \binom{\ell/\lambda}{n}, & \lambda \ne 0; \\ S(n,k), & \lambda = 0, \end{cases} \tag{10}$$

where extended binomial coefficient $\binom{z}{w}$ is defined by

$$\binom{z}{w} = \begin{cases} \dfrac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z \notin \mathbb{N}_-, \quad w, z-w \notin \mathbb{N}_- \\ 0, & z \notin \mathbb{N}_-, \quad w \in \mathbb{N}_- \text{ or } z-w \in \mathbb{N}_- \\ \dfrac{\langle z\rangle_w}{w!}, & z \in \mathbb{N}_-, \quad w \in \mathbb{N}_0 \\ \dfrac{\langle z\rangle_{z-w}}{(z-w)!}, & z, w \in \mathbb{N}_-, \quad z-w \in \mathbb{N}_0 \\ 0, & z, w \in \mathbb{N}_-, \quad z-w \in \mathbb{N}_- \\ \infty, & z \in \mathbb{N}_-, \quad w \notin \mathbb{Z} \end{cases} \tag{11}$$

and the classical Euler gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}.$$

For new results and applications about the Bell polynomials‘ of the second kind $B_{n,k}$, please refer to the papers [13,19,21–23] and closely related references therein.

3 Explicit Formulas of Degenerate λ-Array Type Polynomials

In this section, we establish two explicit formulas for degenerate λ-array type numbers and polynomials, respectively.

Theorem 3.1. For $n \in \mathbb{N}$, degenerate λ-array type numbers $S(n, m; 0; \lambda; \gamma)$ can be computed by

$$S(n, m; 0; \lambda; \gamma) = \frac{n!}{m!} \lambda^n (\lambda - 1)^m \sum_{k=1}^{n} \frac{\langle m\rangle_k}{k!} \frac{1}{(1/\lambda - 1)^k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \binom{\ell/\lambda}{n}, \quad \lambda \neq 0. \tag{12}$$

Proof. Making use of $f(u) = (\lambda u - 1)^m$ and $u = h(t) = (1 + \gamma t)^{1/\gamma} \to 1$ as $t \to 0$ in the Faá di Bruno formula (6) and applying (10) result in

$$\begin{aligned} \frac{d^n\left(\left[\lambda(1+\gamma t)^{1/\gamma} - 1\right]^m\right)}{dt^n} &= \sum_{k=1}^{n} \frac{d^k[(\lambda u - 1)^m]}{du^k} B_{n,k}\left(h'(t), h''(t), \ldots, h^{(k-\ell+1)}(t)\right) \\ &= \sum_{k=1}^{n} \langle m\rangle_k (\lambda u - 1)^{m-k} \lambda^k B_{n,k}\Big((1+\gamma t)^{1/\gamma - 1}, (1-\gamma)(1+\gamma t)^{1/\gamma - 2}, \\ &\quad \ldots, (1-\gamma)(1-2\gamma)\ldots[1-(n-k)\gamma](1+\gamma t)^{1/\gamma - (n-k+1)}\Big) \\ &\to \sum_{k=1}^{n} \langle m\rangle_k (\lambda - 1)^{m-k} \lambda^k B_{n,k}\left(1, 1-\gamma, \ldots, \prod_{\ell=0}^{n-k}(1-\ell\gamma)\right) \\ &= \sum_{k=1}^{n} \langle m\rangle_k (\lambda - 1)^{m-k} \lambda^k (-1)^k \frac{\lambda^n n!}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \binom{\ell/\lambda}{n} \end{aligned}$$

$$=n!\,\lambda^n(\lambda-1)^m\sum_{k=1}^{n}\frac{\langle m\rangle_k}{k!}\frac{1}{(1/\lambda-1)^k}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{\ell/\lambda}{n}$$

as $t\to 0$. Considering the generating function in (5) for $x=0$, we proved the explicit formula (12). The proof of Theorem 3.1 is complete.

Remark 3.1. From (5), it follows immediately that

$$S(0,m;0;\lambda;\gamma)=\frac{(\lambda-1)^m}{m!}.$$

By virtue of the explicit formula (12), we obtain the first few values of degenerate λ-array type numbers $S(n,m;0;\lambda;\gamma)$ for $1\le n\le 6$ as follows:

$$S(1,m;0;\lambda;\gamma)=\frac{1}{(m-1)!}\lambda(\lambda-1)^{m-1},$$

$$S(2,m;0;\lambda;\gamma)=\frac{1}{(m-1)!}\lambda(\lambda-1)^{m-2}[\gamma(1-\lambda)+\lambda m-1],$$

$$S(3,m;0;\lambda;\gamma)=\frac{1}{(m-1)!}\lambda(\lambda-1)^{m-3}\left[2\gamma^2(1-\lambda)^2+3\gamma(1-\lambda)(\lambda m-1)+\lambda^2m^2-3\lambda m+\lambda+1\right],$$

$$\begin{aligned}S(4,m;0;\lambda;\gamma)=\frac{\lambda(\lambda-1)^{m-4}}{(m-1)!}&\left[6\gamma^3(1-\lambda)^3+11\gamma^2(1-\lambda)^2(\lambda m-1)\right.\\&\left.+6\gamma(1-\lambda)\left(\lambda^2m^2-3\lambda m+\lambda+1\right)+\lambda^3m^3-\lambda^2(6m^2-4m+1)+\lambda(7m-4)-1\right],\end{aligned}$$

$$\begin{aligned}S(5,m;0;\lambda;\gamma)=\frac{\lambda(\lambda-1)^{m-5}}{(m-1)!}&\left\{24\gamma^4(\lambda-1)^4-50\gamma^3(\lambda-1)^3(\lambda m-1)\right.\\&+35\gamma^2(\lambda-1)^2\left(\lambda^2m^2-3\lambda m+\lambda+1\right)\\&-10\gamma(\lambda-1)\left[\lambda^3m^3-\lambda^2\left(6m^2-4m+1\right)+\lambda(7m-4)-1\right]\\&\left.+\lambda^4m^4-\lambda^3\left(10m^3-10m^2+5m-1\right)+\lambda^2\left(25m^2-30m+11\right)-\lambda(15m-11)+1\right\},\end{aligned}$$

$$\begin{aligned}S(6,m;0;\lambda;\gamma)=-\frac{\lambda(\lambda-1)^{m-6}}{(m-1)!}&\left\{120\gamma^5(\lambda-1)^5-274\gamma^4(\lambda-1)^4(\lambda m-1)\right.\\&+225\gamma^3(\lambda-1)^3\left(\lambda^2m^2-3\lambda m+\lambda+1\right)\\&-85\gamma^2(\lambda-1)^2\left[\lambda^3m^3-\lambda^2\left(6m^2-4m+1\right)+\lambda(7m-4)-1\right]\\&+15\gamma(\lambda-1)\left[\lambda^4m^4-\lambda^3\left(10m^3-10m^2+5m-1\right)\right.\\&\left.+\lambda^2\left(25m^2-30m+11\right)-\lambda(15m-11)+1\right]-\lambda^5m^5\\&+\lambda^4\left(15m^4-20m^3+15m^2-6m+1\right)-\lambda^3\left(65m^3-120m^2+91m-26\right)\\&\left.+2\lambda^2\left(45m^2-73m+33\right)-\lambda(31m-26)+1\right\}.\end{aligned}$$

Remark 3.2. The explicit formula (12) in Theorem 3.1 and seven concrete values listed in Remark 3.1 reveal that degenerate λ-array type numbers $S(n,m;0;\lambda;\gamma)$ are polynomials of λ and γ with degrees m and $n-1\ge 0$, respectively.

Theorem 3.2. For $n\in\mathbb{N}$, degenerate λ-array type polynomials $S(n,m;x;\lambda;\gamma)$ can be computed by

$$S(n,m;x;\lambda;\gamma)=\frac{n!}{m!}\lambda^n(\lambda-1)^m\sum_{k=1}^{n}\frac{(-1)^k}{k!}\left[\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{\langle m\rangle_\ell\langle x\rangle_{k-\ell}}{(1-1/\lambda)^\ell}\right]\left[\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{\ell/\lambda}{n}\right]. \tag{13}$$

Proof. For $k\in\mathbb{N}$, it is easy to see that

$$\frac{d^k}{du^k}\left[\frac{(\lambda u-1)^m}{m!}u^x\right]=\frac{1}{m!}\sum_{\ell=0}^{k}\binom{k}{\ell}\langle m\rangle_\ell\lambda^\ell(\lambda u-1)^{m-\ell}\langle x\rangle_{k-\ell}u^{x-k+\ell}$$
$$\to\frac{1}{m!}\sum_{\ell=0}^{k}\binom{k}{\ell}\langle m\rangle_\ell\langle x\rangle_{k-\ell}\lambda^\ell(\lambda-1)^{m-\ell}$$

as $u\to 1$. Letting $u=h(t)=(1+\gamma t)^{1/\gamma}\to 1$ as $t\to 0$ and making use of the Faà di Bruno formula (7) give

$$\frac{d^n}{dt^n}\left(\frac{[\lambda(1+\gamma t)^{1/\gamma}-1]^m}{m!}(1+\gamma t)^{x/\gamma}\right)$$
$$=\sum_{k=1}^{n}\frac{d^k}{du^k}\left[\frac{(\lambda u-1)^m}{m!}u^x\right]B_{n,k}\left(h'(t),h''(t),\dots,h^{(k-\ell+1)}(t)\right)$$
$$=\frac{1}{m!}\sum_{k=1}^{n}\left[\sum_{\ell=0}^{k}\binom{k}{\ell}\langle m\rangle_\ell\lambda^\ell(\lambda u-1)^{m-\ell}\langle x\rangle_{k-\ell}u^{x-k+\ell}\right]$$
$$\times B_{n,k}\left((1+\gamma t)^{1/\gamma-1},(1-\gamma)(1+\gamma t)^{1/\gamma-2},\dots,(1+\gamma t)^{1/\gamma-(n-k+1)}\prod_{\ell=0}^{n-k}(1-\ell\gamma)\right)$$
$$\to\frac{1}{m!}\sum_{k=1}^{n}\left[\sum_{\ell=0}^{k}\binom{k}{\ell}\langle m\rangle_\ell\langle x\rangle_{k-\ell}\lambda^\ell(\lambda-1)^{m-\ell}\right]B_{n,k}\left(1,1-\gamma,\dots,\prod_{\ell=0}^{n-k}(1-\ell\gamma)\right)$$
$$=\frac{1}{m!}\sum_{k=1}^{n}\left[\sum_{\ell=0}^{k}\binom{k}{\ell}\langle m\rangle_\ell\langle x\rangle_{k-\ell}\lambda^\ell(\lambda-1)^{m-\ell}\right]\left[(-1)^k\frac{\lambda^n n!}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{\ell/\lambda}{n}\right]$$
$$=\frac{n!}{m!}\lambda^n(\lambda-1)^m\sum_{k=1}^{n}\frac{(-1)^k}{k!}\left[\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{\langle m\rangle_\ell\langle x\rangle_{k-\ell}}{(1-1/\lambda)^\ell}\right]\left[\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{\ell/\lambda}{n}\right]$$

as $t\to 0$. Considering the generating function in (5), we proved the explicit formula (13). The proof of Theorem 3.2 is complete.

Remark 3.3. From the generating function (5), we can easily obtain

$$S(0,m;x;\lambda;\gamma) = \frac{(\lambda-1)^m}{m!}.$$

By virtue of the explicit formula (13), we can calculate the first few values of degenerate λ-array type polynomials $S(n,m;x;\lambda;\gamma)$ for $1 \le n \le 3$ as follows:

$$S(1,m;x;\lambda;\gamma) = \frac{(\lambda-1)^{m-1}}{m!}[\lambda(m+x)-x],$$

$$S(2,m;x;\lambda;\gamma) = \frac{(\lambda-1)^{m-2}}{m!}\left[\lambda^2(m+x)(m-\gamma+x)+\lambda m(\gamma-2x-1)+2\lambda x(\gamma-x)+x(x-\gamma)\right],$$

$$\begin{aligned}S(3,m;x;\lambda;\gamma) = \frac{(\lambda-1)^{m-3}}{m!}&\left\{\lambda^3(m+x)\left[2\gamma^2-3\gamma(m+x)+(m+x)^2\right]\right.\\&+\lambda^2\left[3m^2(\gamma-x-1)-m\left(4\gamma^2-12\gamma x-3\gamma+6x^2+3x-1\right)-3x\left(2\gamma^2-3\gamma x+x^2\right)\right]\\&\left.+\lambda\left[m\left(2\gamma^2-6\gamma x-3\gamma+3x^2+3x+1\right)+3x\left(2\gamma^2-3\gamma x+x^2\right)\right]-x\left(2\gamma^2-3\gamma x+x^2\right)\right\}.\end{aligned}$$

Remark 3.4. From the explicit formula (13) in Theorem 3.2 and the four concrete values in Remark 3.3, we conclude that degenerate λ-array type polynomials $S(n,m;x;\lambda;\gamma)$ are polynomials of x, λ, and γ of degrees n, m, and $n-1 \ge 0$, respectively.

Remark 3.5. When $x=0$ in Theorem 3.2, the explicit formula (13) becomes (12) in Theorem 3.1.

Remark 3.6. For further better understanding degenerate λ-array type polynomials $S(n,m;x;\lambda;\gamma)$, we demonstrate two angles of the graph of $S(3,2;4;\lambda;\gamma)$ for $-9 < \lambda < 9$ and $0 < \gamma < 9$ in Fig. 1. The blue plane in Fig. 1 is the (λ,γ)-plane.

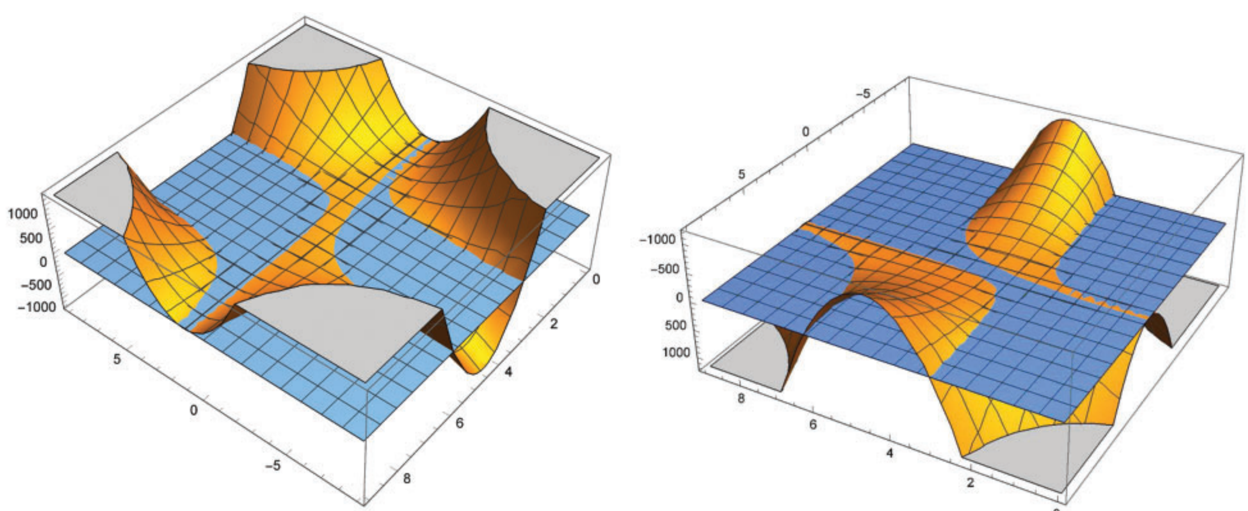

Figure 1: Two angles of the graph of $S(3,2;4;\lambda;\gamma)$ for $-9 < \lambda < 9$ and $0 < \gamma < 9$, plotted by Wolfram Mathematica 12.0

Remark 3.7. For further better understanding degenerate λ-array type polynomials $S(n, m; x; \lambda; \gamma)$, we demonstrate two angles of the graph of $S(3, 2; x; 4; \gamma)$ for $0 < x < 9$ and $0 < \gamma < 9$ in Fig. 2. The blue plane in Fig. 2 is the (x, γ)-plane.

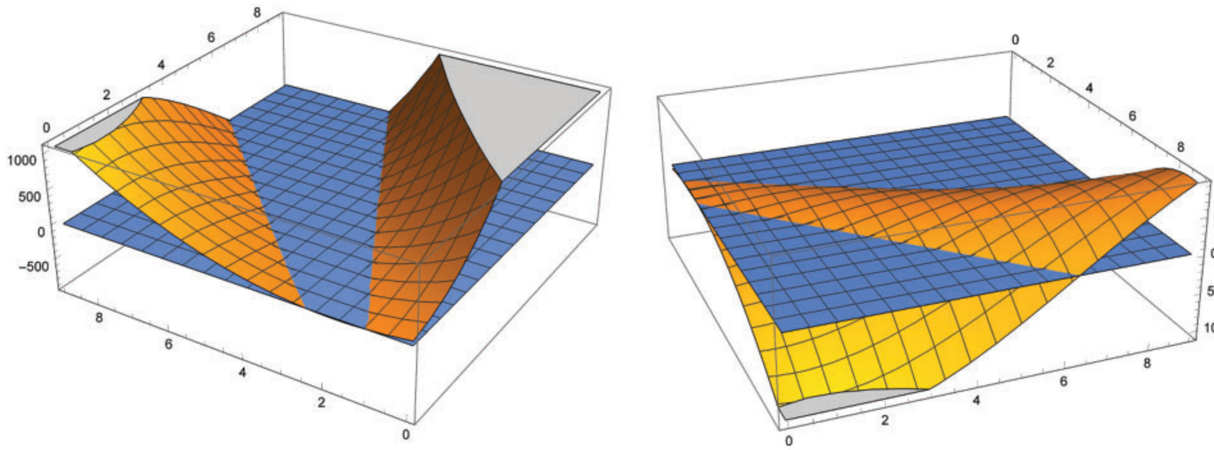

Figure 2: Two angles of the graph of $S(3, 2; x; 4; \gamma)$ for $0 < x < 9$ and $0 < \gamma < 9$, plotted by Wolfram Mathematica 12.0

Remark 3.8. For further better understanding degenerate λ-array type polynomials $S(n, m; x; \lambda; \gamma)$, we demonstrate two angles of the graph of $S(3, 2; x; \lambda; 4)$ for $0 < x < 9$ and $-9 < \lambda < 9$ in Fig. 3. The blue plane in Fig. 3 is the (x, λ)-plane.

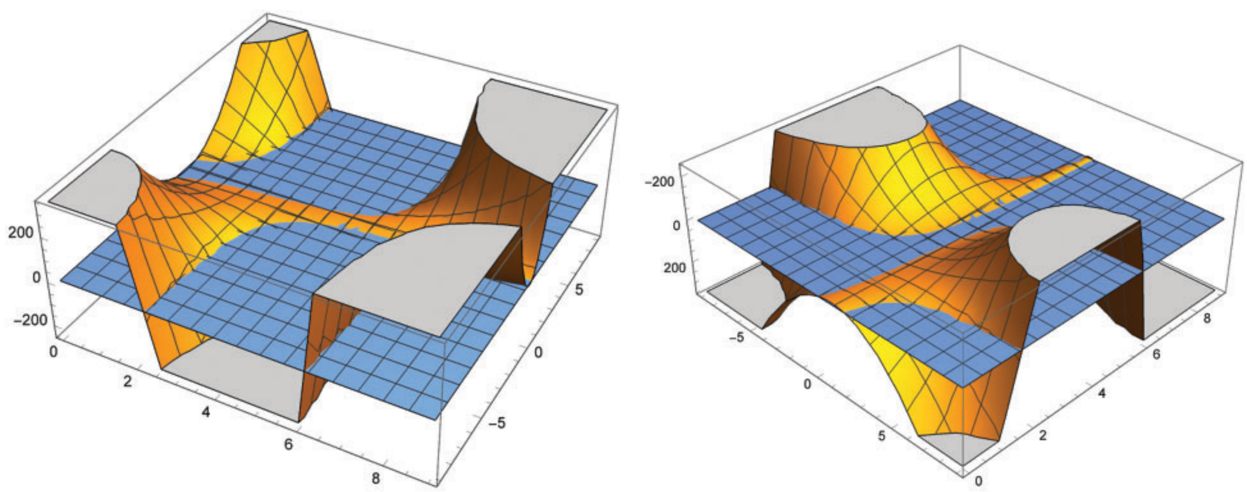

Figure 3: Two angles of the graph of $S(3, 2; x; \lambda; 4)$ for $0 < x < 9$ and $-9 < \lambda < 9$, plotted by Wolfram Mathematica 12.0

4 Recurrence Relations of Degenerate λ-Array Type Polynomials

In this section, we establish several recurrence relations of degenerate λ-array type polynomials.

Theorem 4.1. Degenerate λ-array type polynomials satisfy the recurrence relation

$$\lambda S(n,m;x+1;\lambda;\gamma)=(m+1)S(n,m+1;x;\lambda;\gamma)+S(n,m;x;\lambda;\gamma). \tag{14}$$

Proof. From Eq. (5), it follows that

$$\sum_{n=0}^{\infty}[S(n,m;x+1;\lambda;\gamma)-S(n,m;x;\lambda;\gamma)]\frac{t^n}{n!}$$
$$=\frac{\left[\lambda(1+\gamma t)^{1/\gamma}-1\right]^m}{m!}(1+\gamma t)^{x/\gamma}\left[(1+\gamma t)^{1/\gamma}-1\right]$$
$$=\frac{\left[\lambda(1+\gamma t)^{1/\gamma}-1\right]^{m+1}}{m!}(1+\gamma t)^{x/\gamma}\left[\frac{(1+\gamma)^{1/\gamma}-1}{\lambda(1+\gamma t)^{1/\gamma}-1}\right]$$
$$=\frac{\left[\lambda(1+\gamma t)^{1/\gamma}-1\right]^{m+1}}{m!}(1+\gamma t)^{x/\gamma}\left[\frac{1}{\lambda}+\frac{(1-\lambda)/\lambda}{\lambda(1+\gamma t)^{1/\gamma}-1}\right]$$
$$=\frac{m+1}{\lambda}\frac{\left[\lambda(1+\gamma t)^{1/\gamma}-1\right]^{m+1}}{(m+1)!}(1+\gamma t)^{x/\gamma}+\frac{1-\lambda}{\lambda}\frac{\left[\lambda(1+\gamma t)^{1/\gamma}-1\right]^{m}}{m!}(1+\gamma t)^{x/\gamma}$$
$$=\frac{m+1}{\lambda}\sum_{n=0}^{\infty}S(n,m+1;x;\lambda;\gamma)\frac{t^n}{n!}+\frac{1-\lambda}{\lambda}\sum_{n=0}^{\infty}S(n,m;x;\lambda;\gamma)\frac{t^n}{n!}.$$

Comparing coefficients of the terms $\frac{t^n}{n!}$ on both sides concludes (14). The proof of Theorem 4.1 is complete.

Theorem 4.2. Degenerate λ-array type polynomials $S(n,m;x;\lambda;\gamma)$ satisfy the recurrence relation

$$S(n+1,m;y+\gamma;\lambda;\gamma)=\lambda S(n,m-1;y+1;\lambda;\gamma)+(y+\gamma)S(n,m;y;\lambda;\gamma). \tag{15}$$

Consequently, we have

$$S(n+1,m;y;\lambda)=\lambda S(n,m-1;y+1;\lambda)+yS_2(n,m;y;\lambda). \tag{16}$$

Proof. Differentiating with respect to t yields

$$\frac{d}{dt}\{[\lambda(1+\gamma t)^{1/\gamma}-1]^m(1+\gamma t)^{x/\gamma}\}=\lambda m\left[\lambda(1+\gamma t)^{1/\gamma}-1\right]^{m-1}(1+\gamma t)^{(x+1-\gamma)/\gamma}$$
$$+x\left[\lambda(1+\gamma t)^{1/\gamma}-1\right]^m(1+\gamma t)^{(x-\gamma)/\gamma},$$

where, on the other hand,

$$\frac{d}{dt}\{[\lambda(1+\gamma t)^{1/\gamma}-1]^m(1+\gamma t)^{x/\gamma}\}=m!\sum_{n=0}^{\infty}S(n+1,m;x;\lambda;\gamma)\frac{t^n}{n!},$$

$$x\left[\lambda(1+\gamma t)^{1/\gamma}-1\right]^m(1+\gamma t)^{(x-\gamma)/\gamma}=xm!\sum_{n=0}^{\infty}S(n,m;x-\gamma;\lambda;\gamma)\frac{t^n}{n!},$$

and

$$\lambda m\big[\lambda(1+\gamma t)^{1/\gamma}-1\big]^{m-1}(1+\gamma t)^{(x+1-\gamma)/\gamma}=\lambda m!\sum_{n=0}^{\infty}S(n,m-1;x+1-\gamma;\lambda;\gamma)\frac{t^n}{n!}.$$

Further replacing x by $y+\gamma$ and simplifying lead to (15).

Taking $\gamma\to 0$ in (15) and considering (6) give (16). The proof of Theorem 4.2 is complete.

Remark 4.1. One of anonymous referees commented that the λ-array type polynomials are related to numbers considered in the papers [24–26].

5 Conclusions

In this paper, with the help of the Faá di Bruno formula (7) and the identity (10) for the Bell polynomials of the second kind $B_{n,k}$, we define degenerate λ-array type polynomials $S(n,m;x;\lambda;\gamma)$ by (5), establish two explicit formulas (12) and (13) in Theorems 3.1 and 3.2, and present several recurrence relations (14), (15), and (16) of degenerate λ-array type polynomials and numbers $S(n,m;x;\lambda;\gamma)$ and $S(n,m;\lambda;\gamma)$ in Theorems 4.1 and 4.2.

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