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Neutrosophic $\varkappa$-Structures in Ordered Semigroups

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ABSTRACT

In general, ordered algebraic structures, particularly ordered semigroups, play an important role in fuzzification in many applied areas, such as computer science, formal languages, coding theory, error correction, etc. Nowadays, the concept of ambiguity is important in dealing with a variety of issues related to engineering modeling problems, network theory, decision-making problems in real-life situations, and so on. Several theories have been developed by various researchers to overcome the difficulties that arise from uncertainty, including fuzzy sets, intuitionistic fuzzy sets, probability, soft sets, neutrosophic sets, and many more. In this paper, we focus solely on neutrosophic set theory. In ordered semigroups, we define and investigate the properties of neutrosophic $\varkappa$-ideals and neutrosophic $\varkappa$-interior ideals. We also use neutrosophic $\varkappa$-ideals and neutrosophic $\varkappa$-interior ideals to characterize ordered semigroups.

KEYWORDS

Ordered semigroup; ideals; neutrosophic $\varkappa$-structures; neutrosophic $\varkappa$-ideals; neutrosophic $\varkappa$-interior ideals

1 Introduction

In [1], Zadeh proposed the theory of fuzzy sets to model vague notions in the universe. In [2], Atanassov generalized the fuzzy set theory concepts and renamed as Intuitionistic fuzzy set theory. According to his view, there are the two kinds of degrees of freedom in a globe such as membership to a vague subset and non-membership to that given subset. In [3], Rosenfeld introduced the concepts of fuzziness in groups and obtained several results. Recently, many researchers pursue their research in this area and these concepts have been applied to different algebraic structures such as semigroups, ordered semigroups, rings (see [4–10]).

Smarandache proposed the notions of neutrosophic sets to handle uncertainty that arises everywhere. It is the generalization of fuzzy sets and intuitionistic fuzzy sets. Using these three attributes such as a truth (T), an indeterminacy (I) and a falsity (F) membership functions, neutrosophic sets are characterized. These sets have numerous applications in various disciplines to deal with the complexities that arise primarily from ambiguity data. A neutrosophic set can differentiate between relative and absolute membership functions. Smarandache used these sets in non-standard analysis, namely decision making theory, control theory, decision of sports (winning/defeating/tie), etc.

In [11], Muhiuddin et al. presented the notion of implicative neutrosophic quadruple-algebras, and various properties were investigated. In [12], Muhiuddin et al. found the relationship between $(\varepsilon,\varepsilon)$-neutrosophic ideal and $(\varepsilon,\varepsilon)$-neutrosophic subalgebra in a BCK-algebra. Also, they provided conditions under which an $(\varepsilon,\varepsilon)$-neutrosophic subalgebraic structure to be an $(\varepsilon,\varepsilon)$-neutrosophic ideal structure. In [13], Muhiuddin et al. introduced the theory of neutrosophic implicative $\varkappa$-ideal in BCK-algebras, and examined its properties. In addition, the relationship between different kinds of neutrosophic implicative $\varkappa$-ideals were discussed.

In [14], Khan et al. defined and discussed various properties of neutrosophic $\varkappa$-subsemigroup and ε-neutrosophic $\varkappa$-subsemigroup in a semigroup. As a motivation from [14], we delve into different types of notions of neutrosophic $\varkappa$-structures, namely neutrosophic $\varkappa$-ideals, neutrosophic $\varkappa$-bi-ideals, neutrosophic $\varkappa$-interior ideals and investigated various properties. In [15], Elavarasan et al. proposed the notions of neutrosophic $\varkappa$-ideals and characteristic neutrosophic $\varkappa$-structure in semigroup and discussed its properties. Further, the equivalent assertions for characteristic neutrosophic $\varkappa$-structure were provided. In [16], Porselvi et al. defined and obtained various properties of neutrosophic $\varkappa$-bi-ideal structure in a semigroup. We have shown that both neutrosophic $\varkappa$-bi-ideals and neutrosophic $\varkappa$-right ideals were the same if the semigroup is regular left duo. Moreover, we have obtained equivalent conditions for regular semigroup in terms of neutrosophic $\varkappa$-product. In [17], Porselvi et al. defined and discussed the notions of neutrosophic $\varkappa$-interior ideal structures and neutrosophic $\varkappa$-simple in semigroup. Also, we explored equivalent assertions for a simple semigroup, and neutrosophic $\varkappa$-interior ideal structures. For more concepts related to this work, we refer the readers to [18–23].

The aim of this paper is to define the concepts of neutrosophic $\varkappa$-ideals, neutrosophic $\varkappa$-bi-ideals and neutrosophic $\varkappa$-interior ideals in ordered semigroup, and discuss its properties. We obtain an equivalent assertion of an interior ideal in ordered semigroup in terms of characteristic neutrosophic $\varkappa$-structures. Further, we define the notion of ε-neutrosophic $\varkappa$-ideals and ε-neutrosophic $\varkappa$-interior ideals in ordered semigroup, and explore its properties. Moreover we show that the preimage of neutrosophic $\varkappa$-right (resp., left, ideal) ideal is neutrosophic $\varkappa$-right (resp., left, ideal) ideal under a homomorphism of an ordered semigroup.

2 Preliminaries

An non-empty set $\mathbb{N}$ together with two operations "." and "$\leq$", denoted by $(\mathbb{N},.,\leq)$, is said to be an ordered semigroup if the given assertions are valid:

(i) $(\mathbb{N},.)$ is a semigroup,

(ii) $(\mathbb{N},\leq)$ is a poset,

(iii) For $k_1,k_2 \in \mathbb{N}$, $k_1 \leq k_2 \Rightarrow kk_1 \leq kk_2$ and $k_1k \leq k_2k$ for all $k \in \mathbb{N}$.

For any $A,B \subseteq \mathbb{N}$, $(A] = \{k \in \mathbb{N} : k \leq h$ for some $h \in A\}$ and $AB = \{k_1k_2 :$ for all $k_1 \in A$ and $k_2 \in B\}$.

Definition 2.1. [24] Let $(\mathbb{N},.,\leq)$ be an ordered semigroup and $\phi \neq K \subseteq \mathbb{N}$. Then K is a subsemigroup of $\mathbb{N}$ if $K^2 \subseteq K$.

Definition 2.2. [24] Let $(\mathbb{N},.,\leq)$ be an ordered semigroup. A non-empty subset K in $\mathbb{N}$ is a left (resp., right) ideal of $\mathbb{N}$ if

(i) $\mathbb{N}K \subseteq K$ (resp., $K\mathbb{N} \subseteq K$),

(ii) For $k_1 \in K$ and $k_2 \in \mathbb{N}$; $k_2 \leq k_1 \Rightarrow k_2 \in K$.

K is an ideal of $\mathbb{N}$ if it is a right and a left ideal of $\mathbb{N}$.

Definition 2.3. [24] Let $(\mathbb{N},.,\leq)$ be an ordered semigroup. A subsemigroup K in $\mathbb{N}$ is a bi-ideal of $\mathbb{N}$ if

(i) $K\mathbb{N}K \subseteq K$,

(ii) For all $k_1, k_2 \in \mathbb{N}, k_1 \in K$ and $k_2 \leq k_1$ imply $k_2 \in K$.

Defintion 2.4. [5] Let $(\mathbb{N},.,\leq)$ be an ordered semigroup. $\mathbb{N}$ is called regular if for each $s \in \mathbb{N}, \exists\ k \in \mathbb{N} \ni s \leq sks$.

Defintion 2.5. [5] Let $(\mathbb{N},.,\leq)$ be an ordered semigroup. $\mathbb{N}$ is called intra-regular if for each $s \in \mathbb{N}, \exists\ k_1, k_2 \in \mathbb{N} \ni s \leq k_1 s^2 k_2$.

Let $(\mathbb{N},.,\leq)$ be an ordered semigroup. Then the $\varkappa$-function on $\mathbb{N}$ is a function $h: \mathbb{N} \to [-1,\ 0]$ and the collection of all the $\varkappa$-functions is denoted by $\mathcal{F}(\mathbb{N},[-1,0])$. A $\varkappa$-structure is an ordered pair $(\mathbb{N},h)$ of $\mathbb{N}$ and an $\varkappa$-function h on $\mathbb{N}$.

Definition 2.6. Let $(\mathbb{N},.,\leq)$ be an ordered semigroup. A neutrosophic $\varkappa$-structure in $\mathbb{N}$ is of the form:

$$\mathbb{N}_{\mathcal{P}} := \frac{\mathbb{N}}{(T_{\mathcal{P}}, I_{\mathcal{P}}, F_{\mathcal{P}})} = \left\{ \frac{k}{(T_{\mathcal{P}}(k), I_{\mathcal{P}}(k), F_{\mathcal{P}}(k))} \mid k \in \mathbb{N} \right\},$$

where $T_{\mathcal{P}}, I_{\mathcal{P}}$ and $F_{\mathcal{P}}$ are the negative truth, negative indeterminacy and negative falsity membership functions respectively in $\mathbb{N}$ ($\varkappa$-functions). Obviously $-3 \leq T_{\mathcal{P}}(k) + I_{\mathcal{P}}(k) + F_{\mathcal{P}}(k) \leq 0\ \forall\ k \in \mathbb{N}$.

Definition 2.7. Let $(\mathbb{N},.,\leq)$ be an ordered semigroup. A neutrosophic $\varkappa$-structure $\mathbb{N}_{\mathcal{P}}$ in $\mathbb{N}$ is a neutrosophic $\varkappa$-subsemigroup in $\mathbb{N}$ if the below assertion is valid:

$$(\forall k_1, k_2 \in \mathbb{N}) \quad \begin{pmatrix} T_{\mathcal{P}}(k_1k_2) \leq T_{\mathcal{P}}(k_1) \vee T_{\mathcal{P}}(k_2) \\ I_{\mathcal{P}}(k_1k_2) \geq I_{\mathcal{P}}(k_1) \wedge I_{\mathcal{P}}(k_2) \\ F_{\mathcal{P}}(k_1k_2) \leq F_{\mathcal{P}}(k_1) \vee F_{\mathcal{P}}(k_2) \end{pmatrix}.$$

Let $\mathbb{N}_{\mathcal{P}}$ be a neutrosophic $\varkappa-$structure and $\mu, \delta, \nu \in [-1,0]$. Consider the sets:

$$T_{\mathcal{P}}^{\mu} = \left\{ k \in \mathbb{N} \mid T_{\mathcal{P}}(k) \leq \mu \right\},$$

$$I_{\mathcal{P}}^{\delta} = \left\{ k \in \mathbb{N} \mid I_{\mathcal{P}}(k) \geq \delta \right\},$$

$$F_{\mathcal{P}}^{\nu} = \left\{ k \in \mathbb{N} \mid F_{\mathcal{P}}(k) \leq \nu \right\}.$$

The set $\mathbb{N}_{\mathcal{P}}(\mu,\delta,\nu) := \left\{ k \in \mathbb{N} \mid T_{\mathcal{P}}(k) \leq \mu,\ I_{\mathcal{P}}(k) \geq \delta,\ F_{\mathcal{P}}(k) \leq \nu \right\}$ is known as (μ,δ,ν)-level set on $\mathbb{N}_{\mathcal{P}}$. It is evident that $\mathbb{N}_{\mathcal{P}}(\mu,\delta,\nu) = T_{\mathcal{P}}^{\mu} \cap I_{\mathcal{P}}^{\delta} \cap F_{\mathcal{P}}^{\nu}$.

We refer to [14–17] for basic definitions of neutrosophic $\varkappa$-structures in a semigroup, such as neutrosophic $\varkappa$-ideals, neutrosophic $\varkappa$-bi-ideals, and neutrosophic $\varkappa$-interior ideal. We define the neutrosophic $\varkappa$-structures in an ordered semigroup $\mathbb{N}$ as follows:

Definition 2.8. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. A neutrosophic $\varkappa$-structure $\mathbb{N}_{\mathcal{P}}$ in $\mathbb{N}$ is a neutrosophic $\varkappa$-ideal in $\mathbb{N}$ if the below assertions are valid:

(i) $(\forall k_1, k_2 \in \mathbb{N}) \begin{pmatrix} T_{\mathcal{P}}(k_1k_2) \leq T_{\mathcal{P}}(k_2) \\ I_{\mathcal{P}}(k_1k_2) \geq I_{\mathcal{P}}(k_2) \\ F_{\mathcal{P}}(k_1k_2) \leq F_{\mathcal{P}}(k_2) \end{pmatrix}$.

(ii) $(\forall k_1, k_2 \in \mathbb{N}) \begin{pmatrix} T_{\mathcal{P}}(k_1k_2) \leq T_{\mathcal{P}}(k_1) \\ I_{\mathcal{P}}(k_1k_2) \geq I_{\mathcal{P}}(k_1) \\ F_{\mathcal{P}}(k_1k_2) \leq F_{\mathcal{P}}(k_1) \end{pmatrix}$.

(iii) $(\forall k_1, k_2 \in \mathbb{N}),\ (k_1 \leq k_2) \Rightarrow \begin{pmatrix} T_{\mathcal{P}}(k_1) \leq T_{\mathcal{P}}(k_2) \\ I_{\mathcal{P}}(k_1) \geq I_{\mathcal{P}}(k_2) \\ F_{\mathcal{P}}(k_1) \leq F_{\mathcal{P}}(k_2) \end{pmatrix}$.

A neutrosophic $\varkappa$-structure $\mathbb{N}_{\mathcal{P}}$ in $\mathbb{N}$ is a neutrosophic $\varkappa$-left ideal in $\mathbb{N}$ if the conditions (i) and (iii) are true. A neutrosophic $\varkappa$-structure $\mathbb{N}_{\mathcal{P}}$ in $\mathbb{N}$ is a neutrosophic $\varkappa$-right ideal in $\mathbb{N}$ if the conditions (ii) and (iii) are true.

Definition 2.9. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. A neutrosophic $\varkappa$-subsemigroup $\mathbb{N}_{\mathcal{P}}$ in $\mathbb{N}$ is a neutrosophic $\varkappa$-bi-ideal in $\mathbb{N}$ if the below assertions are valid:

(i) $(\forall a, k_1, k_2 \in \mathbb{N}) \begin{pmatrix} T_{\mathcal{P}}(k_1ak_2) \leq T_{\mathcal{P}}(k_1) \vee T_{\mathcal{P}}(k_2) \\ I_{\mathcal{P}}(k_1ak_2) \geq I_{\mathcal{P}}(k_1) \wedge I_{\mathcal{P}}(k_2) \\ F_{\mathcal{P}}(k_1ak_2) \leq F_{\mathcal{P}}(k_1) \vee F_{\mathcal{P}}(k_2) \end{pmatrix}$.

(ii) $(\forall k_1, k_2 \in \mathbb{N}),\ (k_1 \leq k_2) \Rightarrow \begin{pmatrix} T_{\mathcal{P}}(k_1) \leq T_{\mathcal{P}}(k_2) \\ I_{\mathcal{P}}(k_1) \geq I_{\mathcal{P}}(k_2) \\ F_{\mathcal{P}}(k_1) \leq F_{\mathcal{P}}(k_2) \end{pmatrix}$.

It is evident that all the neutrosophic $\varkappa$-ideals are neutrosophic $\varkappa$-bi-ideals, but neutrosophic $\varkappa$-bi-ideal need not be a neutrosophic $\varkappa$-ideal, as given by an example.

Example 2.1. Consider the ordered semigroup $\mathbb{N} := \{k_1, k_2, k_3, k_4, k_5\}$ with the binary operation "." and the partial order "$\leq$" as follows:

.	k_1	k_2	k_3	k_4	k_5
k_1	k_1	k_4	k_1	k_4	k_4
k_2	k_1	k_2	k_1	k_4	k_4
k_3	k_1	k_4	k_3	k_4	k_5
k_4	k_1	k_4	k_1	k_4	k_4
k_5	k_1	k_4	k_3	k_4	k_5

and $\leq := \{(k_1, k_1), (k_1, k_3), (k_1, k_4), (k_1, k_5), (k_2, k_2), (k_2, k_4), (k_2, k_5), (k_3, k_3), (k_3, k_5), (k_4, k_4), (k_4, k_5), (k_5, k_5)\}$.

Let $\mathbb{N}_{\mathcal{P}} = \left\{ \frac{k_1}{(-0.8,-0.3,-0.7)}, \frac{k_2}{(-0.6,-0.6,-0.3)}, \frac{k_3}{(-0.4,-0.4,-0.5)}, \frac{k_4}{(-0.8,-0.3,-0.7)}, \frac{k_5}{(-0.1,-0.5,-0.3)} \right\}$. Then $\mathbb{N}_{\mathcal{P}}$ is a neutrosophic $\varkappa$-bi-ideal, but not a neutrosophic $\varkappa$-ideal as $T_{\mathbb{N}}(k_3k_5) = -0.1 > T_{\mathbb{N}}(k_3), I_{\mathbb{N}}(k_3k_5) = -0.5 < I_{\mathbb{N}}(k_3)$ and $F_{\mathbb{N}}(k_3k_5) = -0.3 > F_{\mathbb{N}}(k_3)$.

Definition 2.10. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. A neutrosophic $\varkappa$-subsemigroup $\mathbb{N}_{\mathcal{P}}$ in $\mathbb{N}$ is a neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$ if the given assertions are valid:

(i) $(\forall a, k_1, k_2 \in \mathbb{N}) \begin{pmatrix} T_{\mathcal{P}}(k_1ak_2) \leq T_{\mathcal{P}}(a) \\ I_{\mathcal{P}}(k_1ak_2) \geq I_{\mathcal{P}}(a) \\ F_{\mathcal{P}}(k_1ak_2) \leq F_{\mathcal{P}}(a) \end{pmatrix}$.

(ii) $(\forall k_1, k_2 \in \mathbb{N}),\ (k_1 \leq k_2) \Rightarrow \begin{pmatrix} T_{\mathcal{P}}(k_1) \leq T_{\mathcal{P}}(k_2) \\ I_{\mathcal{P}}(k_1) \geq I_{\mathcal{P}}(k_2) \\ F_{\mathcal{P}}(k_1) \leq F_{\mathcal{P}}(k_2) \end{pmatrix}$.

It is evident that neutrosophic $\varkappa$-ideals are always neutrosophic $\varkappa$-interior ideals, but not vice versa, as given by an example.

Example 2.2. Let $\mathbb{N}$ be the set of all non-negative integers except 1. Then $(\mathbb{N}, ., \leq)$ is an ordered semigroup under usual multiplication and the relation $\leq$.

Let $\mathbb{N}_{\mathcal{P}} = \left\{ \frac{0}{(-0.8,-0.1,-0.7)}, \frac{3}{(-0.2,-0.4,-0.5)}, \frac{7}{(-0.5,-0.6,-0.6)}, \frac{21}{(-0.1,-0.5,-0.3)}, \frac{otherwise}{(-0.8,-0.1,-0.7)} \right\}$. Then $\mathbb{N}_{\mathcal{P}}$ is a neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$, but not neutrosophic $\varkappa$-ideal as $T_{\mathbb{N}}(3.7) = -0.1 > T_{\mathbb{N}}(3)$ and $T_{\mathbb{N}}(3.7) = -0.1 > T_{\mathbb{N}}(7)$.

Definition 2.11. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. For any $R \subseteq \mathbb{N}$, the characteristic neutrosophic χ-structure in $\mathbb{N}$ is defined as

$$\chi_R(\mathbb{N}_{\mathcal{P}}) = \frac{\mathbb{N}}{(\chi_R(T)_{\mathcal{P}}, \chi_R(I)_{\mathcal{P}}, \chi_R(F)_{\mathcal{P}})}$$

where

$$\chi_R(T)_{\mathcal{P}} : \mathbb{N} \to [-1, 0],\ k \mapsto \begin{cases} -1 & if\ k \in R \\ 0 & otherwise, \end{cases}$$

$$\chi_R(I)_{\mathcal{P}} : \mathbb{N} \to [-1, 0],\ k \mapsto \begin{cases} 0 & if\ k \in R \\ -1 & otherwise, \end{cases}$$

$$\chi_R(F)_{\mathcal{P}} : \mathbb{N} \to [-1, 0],\ k \mapsto \begin{cases} -1 & if\ k \in R \\ 0 & otherwise. \end{cases}$$

Definition 2.12. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup and let $\mathbb{N}_{\mathcal{K}} := \frac{\mathbb{N}}{(T_{\mathcal{K}}, I_{\mathcal{K}}, F_{\mathcal{K}})}$ and $\mathbb{N}_{\mathcal{P}} := \frac{\mathbb{N}}{(T_{\mathcal{P}}, I_{\mathcal{P}}, F_{\mathcal{P}})}$ be neutrosophic $\varkappa$-structures in $\mathbb{N}$. Then

(i) $\mathbb{N}_{\mathcal{K}}$ is called a neutrosophic $\varkappa$-substructure in $\mathbb{N}_{\mathcal{P}}$, denote by $\mathbb{N}_{\mathcal{P}} \subseteq \mathbb{N}_{\mathcal{K}}$, if $T_{\mathcal{P}}(k) \geq T_{\mathcal{K}}(k), I_{\mathcal{P}}(k) \leq I_{\mathcal{K}}(k), F_{\mathcal{P}}(k) \geq F_{\mathcal{K}}(k)$ for all $k \in \mathbb{N}$.

If $\mathbb{N}_{\mathcal{K}} \subseteq \mathbb{N}_{\mathcal{P}}$ and $\mathbb{N}_{\mathcal{P}} \subseteq \mathbb{N}_{\mathcal{K}}$, then we say that $\mathbb{N}_{\mathcal{K}} = \mathbb{N}_{\mathcal{P}}$.

(ii) The union of two neutrosophic $\varkappa$-structures $\mathbb{N}_{\mathcal{K}}$ and $\mathbb{N}_{\mathcal{P}}$ over $\mathbb{N}$ is defined as

$\mathbb{N}_{\mathcal{K}} \cup \mathbb{N}_{\mathcal{P}} = \mathbb{N}_{\mathcal{K}\cup\mathcal{P}} = (\mathbb{N}; T_{\mathcal{K}\cup\mathcal{P}}, I_{\mathcal{K}\cup\mathcal{P}}, F_{\mathcal{K}\cup\mathcal{P}})$,

where $\forall\ k_1 \in \mathbb{N}$,

$(T_{\mathcal{K}} \cup T_{\mathcal{P}})(k_1) = T_{\mathcal{K}\cup\mathcal{P}}(k_1) = T_{\mathcal{K}}(k_1) \wedge T_{\mathcal{P}}(k_1)$,

$(I_{\mathcal{K}} \cup I_{\mathcal{P}})(k_1) = I_{\mathcal{K}\cap\mathcal{P}}(k_1) = I_{\mathcal{K}}(k_1) \vee I_{\mathcal{P}}(k_1)$,

$(F_{\mathcal{K}} \cup F_{\mathcal{P}})(k_1) = F_{\mathcal{K}\cup\mathcal{P}}(k_1) = F_{\mathcal{K}}(k_1) \wedge F_{\mathcal{P}}(k_1)$.

(iii) The intersection of two neutrosophic $\varkappa$-structures $\mathbb{N}_{\mathcal{K}}$ and $\mathbb{N}_{\mathcal{P}}$ over $\mathbb{N}$ is defined as

$\mathbb{N}_{\mathcal{K}} \cap \mathbb{N}_{\mathcal{P}} = \mathbb{N}_{\mathcal{K}\cap\mathcal{P}} = (\mathbb{N}; T_{\mathcal{K}\cap\mathcal{P}}, I_{\mathcal{K}\cap\mathcal{P}}, F_{\mathcal{K}\cap\mathcal{P}})$,

where $\forall\ k_1 \in \mathbb{N}$,

$(T_{\mathcal{K}} \cap T_{\mathcal{P}})(k_1) = T_{\mathcal{K}\cap\mathcal{P}}(k_1) = T_{\mathcal{K}}(k_1) \vee T_{\mathcal{P}}(k_1)$,
$(I_{\mathcal{K}} \cap I_{\mathcal{P}})(k_1) = I_{\mathcal{K}\cap\mathcal{P}}(k_1) = I_{\mathcal{K}}(k_1) \wedge I_{\mathcal{P}}(k_1)$,
$(F_{\mathcal{K}} \cap F_{\mathcal{P}})(k_1) = F_{\mathcal{K}\cap\mathcal{P}}(k_1) = F_{\mathcal{K}}(k_1) \vee F_{\mathcal{P}}(k_1)$.

3 Neutrosophic $\varkappa$-Structures in Ordered Semigroups

In this section, we study some properties of neutrosophic $\varkappa$-ideals and neutrosophic $\varkappa$-interior ideal structure in an ordered semigroup $\mathbb{N}$. It is evident that neutrosophic $\varkappa$-ideals are neutrosophic $\varkappa$-interior ideals in $\mathbb{N}$, but the converse part need not be true in general. Further, we show that all neutrosophic $\varkappa$-interior ideals are neutrosophic $\varkappa$-ideals under certain conditions. Unless otherwise stated, we assume that $\mathbb{N}_{\mathcal{P}}$ and $\mathbb{N}_{\mathcal{K}}$ are neutrosophic $\varkappa$-structures in $\mathbb{N}$ throughout this section.

Theorem 3.1. [15] Let $\mathbb{N}$ be a semigroup. Then for any $K \subseteq \mathbb{N}$, the given assertions are equivalent:

(i) K is left ideal (resp., right ideal),

(ii) $\chi_K(\mathbb{N}_{\mathcal{K}})$ is neutrosophic $\varkappa$-left ideal(resp., right ideal).

Theorem 3.2. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. Then for any $K \subseteq \mathbb{N}$, the given assertions are equivalent:

(i) K is left ideal (resp., right ideal),

(ii) $\chi_K(\mathbb{N}_{\mathcal{K}})$ is neutrosophic $\varkappa$-left ideal(resp., right ideal).

Proof. (i) $\Rightarrow$ (ii) Suppose K is left ideal and let $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq k_2$. Then $T_{\mathcal{K}}(k_1) \leq T_{\mathcal{K}}(k_2)$, $I_{\mathcal{K}}(k_1) \geq I_{\mathcal{K}}(k_2)$ and $F_{\mathcal{K}}(k_1) \leq F_{\mathcal{K}}(k_2)$.

If $k_2 \in K$, then $\chi_K(T)_{\mathcal{K}}(k_2) = -1$, $\chi_K(I)_{\mathcal{K}}(k_2) = 0$ and $\chi_K(F)_{\mathcal{K}}(k_2) = -1$ and so $\chi_K(T)_{\mathcal{K}}(k_1) \leq \chi_K(T)_{\mathcal{K}}(k_2) = -1$, $\chi_K(I)_{\mathcal{K}}(k_1) \geq \chi_K(I)_{\mathcal{K}}(k_2) = 0$ and $\chi_K(F)_{\mathcal{K}}(k_1) \leq \chi_K(F)_{\mathcal{K}}(k_2) = -1$.

If $k_2 \notin K$, then $\chi_K(T)_{\mathcal{K}}(k_2) = 0$, $\chi_K(I)_{\mathcal{K}}(k_2) = -1$ and $\chi_K(F)_{\mathcal{K}}(k_2) = 0$ and so $\chi_K(T)_{\mathcal{K}}(k_1) \leq 0 = \chi_K(T)_{\mathcal{K}}(k_2)$, $\chi_K(I)_{\mathcal{K}}(k_1) \geq -1 = \chi_K(I)_{\mathcal{K}}(k_2)$ and $\chi_K(F)_{\mathcal{K}}(k_1) \leq 0 = \chi_K(F)_{\mathcal{K}}(k_2)$. Thus $\chi_K(\mathbb{N}_{\mathcal{K}})$ is neutrosophic $\varkappa$-left ideal. By Theorem 3.1, $\chi_K(\mathbb{N}_{\mathcal{K}})$ is neutrosophic $\varkappa$-left ideal.

(ii) $\Rightarrow$ (i) Assume $\chi_K(\mathbb{N}_{\mathcal{K}})$ is neutrosophic $\varkappa$-left ideal. Let $k_1 \in K$, $k_2 \in \mathbb{N}$ and $k_2 \leq k_1$. Then $\chi_K(T)_{\mathcal{K}}(k_2) \leq \chi_K(T)_{\mathcal{K}}(k_1) = -1$, $\chi_K(I)_{\mathcal{K}}(k_2) \geq \chi_K(I)_{\mathcal{K}}(k_1) = 0$ and $\chi_K(F)_{\mathcal{K}}(k_2) \leq \chi_K(F)_{\mathcal{K}}(k_1) = -1$ which imply $k_2 \in K$. Hence by Theorem 3.1, K is left ideal.

Theorem 3.3. [17] Suppose $\mathbb{N}$ is a semigroup and for any $K \subseteq \mathbb{N}$, the equivalent assertions are:

(i) K is interior ideal,

(ii) $\chi_K(\mathbb{N}_{\mathcal{K}})$ is neutrosophic $\varkappa$-interior ideal.

Theorem 3.4. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. Then for any $K \subseteq \mathbb{N}$, the given assertions are equivalent:

(i) K is interior ideal,

(ii) $\chi_K(\mathbb{N}_{\mathcal{K}})$ is neutrosophic $\varkappa$-interior ideal.

Proof. (i) $\Rightarrow$ (ii) Suppose K is interior ideal and let $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq k_2$. Then $T_{\mathcal{K}}(k_1) \leq T_{\mathcal{K}}(k_2)$, $I_{\mathcal{K}}(k_1) \geq I_{\mathcal{K}}(k_2)$ and $F_{\mathcal{K}}(k_1) \leq F_{\mathcal{K}}(k_2)$.

If $k_2 \in K$, then $\chi_K(T)_{\mathcal{K}}(k_2) = -1$, $\chi_K(I)_{\mathcal{K}}(k_2) = 0$ and $\chi_K(F)_{\mathcal{K}}(k_2) = -1$ and so $\chi_K(T)_{\mathcal{K}}(k_1) = -1$, $\chi_K(I)_{\mathcal{K}}(k_1) = 0$ and $\chi_K(F)_{\mathcal{K}}(k_1) = -1$.

If $k_2 \notin K$, then $\chi_K(T)_{\mathcal{K}}(k_2) = 0$, $\chi_K(I)_{\mathcal{K}}(k_2) = -1$ and $\chi_K(F)_{\mathcal{K}}(k_2) = 0$ and so $\chi_K(T)_{\mathcal{K}}(k_1) \leq \chi_K(T)_{\mathcal{K}}(k_2) = 0$, $\chi_K(I)_{\mathcal{K}}(k_1) \geq \chi_K(I)_{\mathcal{K}}(k_2) = -1$ and $\chi_K(F)_{\mathcal{K}}(k_1) \leq \chi_K(F)_{\mathcal{K}}(k_2) = 0$. Thus $\chi_K(\mathbb{N}_{\mathcal{K}})$ is neutrosophic $\varkappa$-interior ideal by Theorem 3.3.

(ii) $\Rightarrow$ (i) Assume $\chi_K(\mathbb{N}_{\mathcal{K}})$ is neutrosophic $\varkappa$-interior ideal. Let $k_1 \in K$, $k_2 \in \mathbb{N}$ and $k_2 \leq k_1$. Then $\chi_K(T)_{\mathcal{K}}(k_2) \leq \chi_K(T)_{\mathcal{K}}(k_1) = -1$, $\chi_K(I)_{\mathcal{K}}(k_2) \geq \chi_K(I)_{\mathcal{K}}(k_1) = 0$ and $\chi_K(F)_{\mathcal{K}}(k_2) \leq \chi_K(F)_{\mathcal{K}}(k_1) = -1$ which imply $k_2 \in K$. By Theorem 3.3, K is interior ideal.

Theorem 3.5. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. Then the arbitrary intersection (resp., union) of neutrosophic $\varkappa$-interior ideals in $\mathbb{N}$ is a neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$.

Proof. The proof is a routine procedure.

Theorem 3.6. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. If $\mathbb{N}$ is regular, then neutrosophic $\varkappa$-interior ideals in $\mathbb{N}$ are neutrosophic $\varkappa$-ideals.

Proof. Assume $\mathbb{N}_{\mathcal{P}}$ is neutrosophic $\varkappa$-interior ideal and let $k_1, k_2 \in \mathbb{N}$. As $k_1 \in \mathbb{N}$ and $\mathbb{N}$ is regular, there is $r \in \mathbb{N}$ such that $k_1 \leq k_1 r k_1$. Now, $T_{\mathcal{P}}(k_2 k_1) \leq T_{\mathcal{P}}(k_2 k_1 r k_1) \leq T_{\mathcal{P}}(k_1), I_{\mathcal{P}}(k_2 k_1) \geq I_{\mathcal{P}}(k_2 k_1 r k_1) \geq I_{\mathcal{P}}(k_1)$ and $F_{\mathcal{P}}(k_2 k_1) \leq F_{\mathcal{P}}(k_2 k_1 r k_1) \leq F_{\mathcal{P}}(k_1)$. Therefore $\mathbb{N}_{\mathcal{P}}$ is neutrosophic $\varkappa$-left ideal. In a similar way, we can claim that $\mathbb{N}_{\mathcal{P}}$ is neutrosophic $\varkappa$-right ideal.

Theorem 3.7. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. If $\mathbb{N}$ is intra-regular, then neutrosophic $\varkappa$-interior ideals in $\mathbb{N}$ are neutrosophic $\varkappa$-ideals.

Proof. Let $\mathbb{N}_{\mathcal{P}}$ be neutrosophic $\varkappa$-interior ideal and $k_1, k_2 \in \mathbb{N}$. As $k_1 \in \mathbb{N}$ and $\mathbb{N}$ is intra regular, $\exists\ s, t \in \mathbb{N} \ni k_1 \leq s k_1^2 t$. Now,

$$T_{\mathcal{P}}(k_2 k_1) \leq T_{\mathcal{P}}(k_2 s k_1^2 t) \leq T_{\mathcal{P}}(k_1),$$

$$I_{\mathcal{P}}(k_2 k_1) \geq I_{\mathcal{P}}(k_2 s k_1^2 t) \geq I_{\mathcal{P}}(k_1),$$

$$F_{\mathcal{P}}(k_2 k_1) \leq F_{\mathcal{P}}(k_2 s k_1^2 t) \leq F_{\mathcal{P}}(k_1).$$

Therefore $\mathbb{N}_{\mathcal{P}}$ is neutrosophic $\varkappa$-left ideal. In the same way, we can claim that $\mathbb{N}_{\mathcal{P}}$ is neutrosophic $\varkappa$-right ideal and hence $\mathbb{N}_{\mathcal{P}}$ is neutrosophic $\varkappa$-ideal.

Definition 3.1. An ordered semigroup $\mathbb{N}$ is said to be

(i) left (resp., right) simple if it does not contain any proper left (resp., right) ideal of $\mathbb{N}$.

(ii) simple if it does not contain any proper ideal of $\mathbb{N}$.

Definition 3.2. An ordered semigroup $\mathbb{N}$ is known as neutrosophic $\varkappa$-simple if all the neutrosophic $\varkappa$-ideals are constant functions i.e., for any neutrosophic $\varkappa$-ideal $\mathbb{N}_{\mathcal{P}}$ in $\mathbb{N}$, we can have $T_{\mathcal{P}}(k_1) = T_{\mathcal{P}}(k_2)$, $I_{\mathcal{P}}(k_1) = I_{\mathcal{P}}(k_2)$ and $F_{\mathcal{P}}(k_1) = F_{\mathcal{P}}(k_2)$ for all $k_1, k_2 \in \mathbb{N}$.

Notation 3.1. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. Then for any $k \in \mathbb{N}$, we define $J_k \subseteq \mathbb{N}$ as follows:

$$J_k := \left\{ m \in \mathbb{N} \mid T_{\mathcal{K}}(m) \leq T_{\mathcal{K}}(k),\ I_{\mathcal{K}}(m) \geq I_{\mathcal{K}}(k) \quad \text{and} \quad F_{\mathcal{K}}(m) \leq F_{\mathcal{K}}(k) \right\}.$$

Theorem 3.8. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. If $\mathbb{N}_{\mathcal{K}}$ is a neutrosophic $\varkappa$-right (resp., $\varkappa$-left, $\varkappa$-ideal) ideal in $\mathbb{N}$, then for any $k \in \mathbb{N}, J_k$ is a right ideal (resp., left ideal, ideal) of $\mathbb{N}$.

Proof. Let $k \in \mathbb{N}$. Then clearly $\phi \neq J_k \subseteq \mathbb{N}$. Let $k_1 \in J_k$ and $k_2 \in \mathbb{N}$. Then $k_1k_2 \in J_k$. Indeed; Since $k_1, k_2 \in \mathbb{N}$ and $\mathbb{N}_{\mathcal{K}}$ is neutrosophic $\varkappa$-right ideal, we get $T_{\mathcal{K}}(k_1k_2) \leq T_{\mathcal{K}}(k_1)$, $I_{\mathcal{K}}(k_1k_2) \geq I_{\mathcal{K}}(k_1)$ and $F_{\mathcal{K}}(k_1k_2) \leq F_{\mathcal{K}}(k_1)$. Since $k_1 \in J_k$, we get $T_{\mathcal{K}}(k_1) \leq T_{\mathcal{K}}(k)$, $I_{\mathcal{K}}(k_1) \geq I_{\mathcal{K}}(k)$ and $F_{\mathcal{K}}(k_1) \leq F_{\mathcal{K}}(k)$ which imply $k_1k_2 \in J_k$. Let $a_1 \in J_k$, $a_2 \in \mathbb{N}$ with $a_2 \leq a_1$. Then $T_{\mathcal{K}}(a_2) \leq T_{\mathcal{K}}(a_1)$, $I_{\mathcal{K}}(a_2) \geq I_{\mathcal{K}}(a_1)$ and $F_{\mathcal{K}}(a_2) \leq F_{\mathcal{K}}(a_1)$. Since $a_1 \in J_k$, we have $T_{\mathcal{K}}(a_1) \leq T_{\mathcal{K}}(k)$, $I_{\mathcal{K}}(a_1) \geq I_{\mathcal{K}}(k)$ and $F_{\mathcal{K}}(a_1) \leq F_{\mathcal{K}}(k)$. So $T_{\mathcal{K}}(a_2) \leq T_{\mathcal{K}}(k), I_{\mathcal{K}}(a_2) \geq I_{\mathcal{K}}(k)$ and $F_{\mathcal{K}}(a_2) \leq F_{\mathcal{K}}(k)$ which imply $a_2 \in J_k$. Therefore J_k is a right ideal in $\mathbb{N}$.

Theorem 3.9. If $\mathbb{N}$ is an ordered semigroup, then $\mathbb{N}$ is neutrosophic $\varkappa$-simple if and only if $\mathbb{N}$ is simple.

Proof. Suppose $\mathbb{N}$ is neutrosophic $\varkappa$-simple. Let J be an ideal in $\mathbb{N}$. Then by Theorem 3.2, $\chi_J(\mathbb{N}_{\mathcal{K}})$ is neutrosophic $\varkappa$-ideal. We now prove that $\mathbb{N} = J$. Let $k \in \mathbb{N}$. Since $\mathbb{N}$ is neutrosophic $\varkappa$-simple, $\chi_J(\mathbb{N}_{\mathcal{K}})$ is constant and $\chi_J(\mathbb{N}_{\mathcal{K}})(k) = \chi_J(\mathbb{N}_{\mathcal{K}})(k')$ for every $k' \in \mathbb{N}$. In particular, we have $\chi_J(T_{\mathcal{K}})(k) = \chi_J(T_{\mathcal{K}})(d) = -1$, $\chi_J(I_{\mathcal{K}})(k) = \chi_J(I_{\mathcal{K}})(d) = 0$ and $\chi_J(F_{\mathcal{K}})(k) = \chi_J(F_{\mathcal{K}})(d) = -1$ for any $d \in J$ which gives $k \in J$. Thus $\mathbb{N} \subseteq J$ and hence $\mathbb{N} = J$.

Conversely, let $\mathbb{N}_{\mathcal{K}}$ be neutrosophic $\varkappa$-ideal with $k_1, k_2 \in \mathbb{N}$. Then by Theorem 3.8, J_{k_1} is an ideal. As $\mathbb{N}$ is simple, we have $J_{k_1} = \mathbb{N}$. Since $k_2 \in J_{k_1}$, we have $T_{\mathcal{K}}(k_2) \leq T_{\mathcal{K}}(k_1)$, $I_{\mathcal{K}}(k_2) \geq I_{\mathcal{K}}(k_1)$ and $F_{\mathcal{K}}(k_2) \leq F_{\mathcal{K}}(k_1)$. Similarly, we can prove that $T_{\mathcal{K}}(k_1) \leq T_{\mathcal{K}}(k_2)$, $I_{\mathcal{K}}(k_1) \geq I_{\mathcal{K}}(k_2)$ and $F_{\mathcal{K}}(k_1) \leq F_{\mathcal{K}}(k_2)$. So $T_{\mathcal{K}}(k_2) = T_{\mathcal{K}}(k_1)$, $I_{\mathcal{K}}(k_2) = I_{\mathcal{K}}(k_1)$ and $F_{\mathcal{K}}(k_2) = F_{\mathcal{K}}(k_1)$. Hence $\mathbb{N}$ is neutrosophic $\varkappa$-simple.

Lemma 3.1. [25] An ordered semigroup $\mathbb{N}$ is simple if and only if $\mathbb{N} = (\mathbb{N}t\mathbb{N}]$ for all $t \in \mathbb{N}$.

Theorem 3.10. For any ordered semigroup $\mathbb{N}, \mathbb{N}$ is simple if and only if all the neutrosophic $\varkappa$-interior ideals in $\mathbb{N}$ are constant functions.

Proof. Suppose $k_1, k_2 \in \mathbb{N}$ and $\mathbb{N}$ is simple. Let $\mathbb{N}_{\mathcal{K}}$ be neutrosophic $\varkappa$-interior ideal. Then by Lemma 3.1, we get $\mathbb{N} = (\mathbb{N}k_1\mathbb{N}] = (\mathbb{N}k_2\mathbb{N}]$. Since $k_1 \in (\mathbb{N}k_1\mathbb{N}]$, we get $k_1 \leq tk_2s$ for $t, s \in \mathbb{N}$. Since $\mathbb{N}_{\mathcal{K}}$ is neutrosophic $\varkappa$-interior ideal, we can have $T_{\mathcal{K}}(k_1) \leq T_{\mathcal{K}}(tk_2s) \leq T_{\mathcal{K}}(k_2), I_{\mathcal{K}}(k_1) \geq I_{\mathcal{K}}(tk_2s) \geq I_{\mathcal{K}}(k_2)$ and $F_{\mathcal{K}}(k_1) \leq F_{\mathcal{K}}(tk_2s) \leq F_{\mathcal{K}}(k_2)$. Similarly, we can prove that $T_{\mathcal{K}}(k_2) \leq T_{\mathcal{K}}(k_1)$, $I_{\mathcal{K}}(k_2) \geq I_{\mathcal{K}}(k_1)$ and $F_{\mathcal{K}}(k_2) \leq F_{\mathcal{K}}(k_1)$. So $\mathbb{N}_{\mathcal{K}}$ is constant.

Conversely, suppose $\mathbb{N}_{\mathcal{K}}$ is a neutrosophic $\varkappa$-ideal in $\mathbb{N}$. Then $\mathbb{N}_{\mathcal{K}}$ is neutrosophic $\varkappa$-interior ideal. By assumption, $\mathbb{N}_{\mathcal{K}}$ is constant and hence $\mathbb{N}_{\mathcal{K}}$ is neutrosophic $\varkappa$-simple. Therefore $\mathbb{N}$ is simple, by Theorem 3.9.

Theorem 3.11. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. If $\mathbb{N}_{\mathcal{P}}$ is neutrosophic $\varkappa$-interior ideal with $\mu, \delta, \nu \in [-1, 0]$ and $-3 \leq \mu + \delta + \nu \leq 0$, then (μ, δ, ν)-level set in $\mathbb{N}_{\mathcal{P}}$ is neutrosophic $\varkappa$-interior ideal provided $\mathbb{N}_{\mathcal{P}}(\mu, \delta, \nu) \neq \phi$.

Proof. Suppose $\mathbb{N}_{\mathcal{P}}(\mu, \delta, \nu) \neq \phi$ for $\mu, \delta, \nu \in [-1, 0]$. Let $\mathbb{N}_{\mathcal{P}}$ be neutrosophic $\varkappa$-interior ideal and $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq k_2$. Then $T_{\mathcal{P}}(k_1) \leq T_{\mathcal{P}}(k_2)$, $I_{\mathcal{P}}(k_1) \geq I_{\mathcal{P}}(k_2)$ and $F_{\mathcal{P}}(k_1) \leq F_{\mathcal{P}}(k_2)$. If $k_2 \in T^{\mu}_{\mathcal{P}}$, then $T_{\mathcal{P}}(k_2) \leq \mu$ which implies $T_{\mathcal{P}}(k_1) \leq \mu$ and so $k_1 \in T^{\mu}_{\mathcal{P}}$. If $k_2 \in I^{\delta}_{\mathcal{P}}$, then $I_{\mathcal{P}}(k_2) \geq \delta$ which implies $I_{\mathcal{P}}(k_1) \geq \delta$ and so $k_1 \in I^{\delta}_{\mathcal{P}}$. If $k_2 \in F^{\nu}_{\mathcal{P}}$, then $F_{\mathcal{P}}(k_2) \leq \nu$ which implies $F_{\mathcal{P}}(k_1) \leq \nu$ and so $k_1 \in F^{\nu}_{\mathcal{P}}$. Hence $k_1 \in \mathbb{N}_{\mathcal{P}}(\mu, \delta, \nu)$. By [[17], Theorem 3.16], $\mathbb{N}_{\mathcal{P}}(\mu, \delta, \nu)$ is a neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$.

Theorem 3.12. ([17], Theorem 3.17) Let $\mathbb{N}$ be a semigroup and $\mathbb{N}_{\mathcal{K}}$ be neutrosophic $\varkappa$-structure in $\mathbb{N}$ with $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $T^{\alpha}_{\mathcal{K}}, I^{\beta}_{\mathcal{K}}$ and $F^{\gamma}_{\mathcal{K}}$ are interior ideals in $\mathbb{N}$, then $\mathbb{N}_{\mathcal{K}}$ is neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$ whenever $T^{\alpha}_{\mathcal{K}} \neq \phi$, $I^{\beta}_{\mathcal{K}} \neq \phi$ and $F^{\gamma}_{\mathcal{K}} \neq \phi$.

Theorem 3.13. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. Let $\mathbb{N}_{\mathcal{K}}$ be neutrosophic $\varkappa$-structure with $\alpha, \beta, \gamma \in [-1, 0]$ and $-3 \leq \alpha + \beta + \gamma \leq 0$. If $T^{\alpha}_{\mathcal{K}}, I^{\beta}_{\mathcal{K}}$ and $F^{\gamma}_{\mathcal{K}}$ are interior ideals, then $\mathbb{N}_{\mathcal{K}}$ is neutrosophic $\varkappa$-interior ideal whenever $T^{\alpha}_{\mathcal{K}} \neq \phi$, $I^{\beta}_{\mathcal{K}} \neq \phi$ and $F^{\gamma}_{\mathcal{K}} \neq \phi$.

Proof. Let $k_1 \in \mathbb{N}$, $k_2 \in T^{\alpha}_{\mathcal{K}}$ with $k_1 \leq k_2$. Then $k_1 \in T^{\alpha}_{\mathcal{K}}$ as $T^{\alpha}_{\mathcal{K}}$ is an interior ideal of $\mathbb{N}$. Suppose $T_{\mathcal{K}}(k_1) > T_{\mathcal{K}}(k_2)$. Then $T_{\mathcal{K}}(k_1) > t_{\alpha} \geq T_{\mathcal{K}}(k_2)$ for some $t_{\alpha} \in [-1, 0)$. So $k_2 \in T^{t_{\alpha}}_{\mathcal{K}}$ but $k_1 \notin T^{t_{\alpha}}_{\mathcal{K}}$, a contradiction. Thus $T_{\mathcal{K}}(k_1) \leq T_{\mathcal{K}}(k_2)$.

Let $k_1 \in \mathbb{N}$, $k_2 \in I^{\beta}_{\mathcal{K}}$ with $k_1 \leq k_2$. Then $k_1 \in I^{\beta}_{\mathcal{K}}$ as $I^{\beta}_{\mathcal{K}}$ is an interior ideal of $\mathbb{N}$. Suppose $I_{\mathcal{K}}(k_1) < I_{\mathcal{K}}(k_2)$. Then $I_{\mathcal{K}}(k_1) < t_{\beta} \leq I_{\mathcal{K}}(k_2)$ for some $t_{\beta} \in [-1, 0)$. So $k_2 \in I^{t_{\beta}}_{\mathcal{K}}$ but $k_1 \notin I^{t_{\beta}}_{\mathcal{K}}$, a contradiction. Thus $I_{\mathcal{K}}(k_1) \geq I_{\mathcal{K}}(k_2)$.

Let $k_1 \in \mathbb{N}$, $k_2 \in F^{\gamma}_{\mathcal{K}}$ with $k_1 \leq k_2$. Then $k_1 \in F^{\gamma}_{\mathcal{K}}$ as $F^{\gamma}_{\mathcal{K}}$ is an interior ideal of $\mathbb{N}$. Suppose $F_{\mathcal{K}}(k_1) > F_{\mathcal{K}}(k_2)$. Then $F_{\mathcal{K}}(k_1) > t_{\gamma} \geq F_{\mathcal{K}}(k_2)$ for some $t_{\gamma} \in [-1, 0)$. So $k_2 \in F^{t_{\gamma}}_{\mathcal{K}}$ but $k_1 \notin F^{t_{\gamma}}_{\mathcal{K}}$, a contradiction. Thus $F_{\mathcal{K}}(k_1) \leq F_{\mathcal{K}}(k_2)$.

Hence by Theorem 3.12, $\mathbb{N}_{\mathcal{K}}$ is neutrosophic $\varkappa$-interior ideal.

Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. Following [14], we define ε-neutrosophic $\varkappa$-subsemigroup, ε-neutrosophic $\varkappa$-ideal and ε-neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$ as follows:

Definition 3.3. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. A neutrosophic $\varkappa$-structure $\mathbb{N}_{\mathcal{P}}$ in $\mathbb{N}$ is called an ε-neutrosophic $\varkappa$-subsemigroup in $\mathbb{N}$ if the given assertion is valid:

$$(\forall k_1, k_2 \in \mathbb{N}) \begin{pmatrix} T_{\mathcal{P}}(k_1k_2) \leq \vee\{T_{\mathcal{P}}(k_1), T_{\mathcal{P}}(k_2), \varepsilon_T\} \\ I_{\mathcal{P}}(k_1k_2) \geq \wedge\{I_{\mathcal{P}}(k_1), I_{\mathcal{P}}(k_2), \varepsilon_I\} \\ F_{\mathcal{P}}(k_1k_2) \leq \vee\{F_{\mathcal{P}}(k_1), F_{\mathcal{P}}(k_2), \varepsilon_F\} \end{pmatrix}$$

where $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [-1, 0]$ such that $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$.

Definition 3.4. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. A neutrosophic $\varkappa$-structure $\mathbb{N}_{\mathcal{P}}$ in $\mathbb{N}$ is called an ε-neutrosophic $\varkappa$-ideal of $\mathbb{N}$ if the given assertions are valid:

(i) $$(\forall k_1, k_2 \in \mathbb{N}) \begin{pmatrix} T_{\mathcal{P}}(k_1k_2) \leq \vee\{T_{\mathcal{P}}(k_2), \varepsilon_T\} \\ I_{\mathcal{P}}(k_1k_2) \geq \wedge\{I_{\mathcal{P}}(k_2), \varepsilon_I\} \\ F_{\mathcal{P}}(k_1k_2) \leq \vee\{F_{\mathcal{P}}(k_2), \varepsilon_F\} \end{pmatrix}.$$

(ii) $(\forall k_1, k_2 \in \mathbb{N}) \begin{pmatrix} T_{\mathcal{P}}(k_1k_2) \leq \vee\{T_{\mathcal{P}}(k_1), \varepsilon_T\} \\ I_{\mathcal{P}}(k_1k_2) \geq \wedge\{I_{\mathcal{P}}(k_1), \varepsilon_I\} \\ F_{\mathcal{P}}(k_1k_2) \leq \vee\{F_{\mathcal{P}}(k_1), \varepsilon_F\} \end{pmatrix}$.

(iii) $(\forall k_1, k_2 \in \mathbb{N}),\ k_1 \leq k_2 \Rightarrow \begin{pmatrix} T_{\mathcal{P}}(k_1) \leq T_{\mathcal{P}}(k_2) \vee \varepsilon_T \\ I_{\mathcal{P}}(k_1) \geq I_{\mathcal{P}}(k_2) \wedge \varepsilon_I \\ F_{\mathcal{P}}(k_1) \leq F_{\mathcal{P}}(k_2) \vee \varepsilon_F \end{pmatrix}$

where $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [-1, 0]$ such that $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$.

$\mathbb{N}_{\mathcal{P}}$ is called an ε-neutrosophic $\varkappa$-left ideal of $\mathbb{N}$ if it satisfies the assertions (i) and (iii).

$\mathbb{N}_{\mathcal{P}}$ is called an ε-neutrosophic $\varkappa$-right ideal of $\mathbb{N}$ if it satisfies the assertions (ii) and (iii).

Definition 3.5. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. A neutrosophic $\varkappa$-subsemigroup $\mathbb{N}_{\mathcal{P}}$ in $\mathbb{N}$ is called a ε-neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$ if the following assertions are valid:

(i) $(\forall k_1, k_2, k_3 \in \mathbb{N}) \begin{pmatrix} T_{\mathcal{P}}(k_1k_2k_3) \leq \vee\{T_{\mathcal{P}}(k_2), \varepsilon_T\} \\ I_{\mathcal{P}}(k_1k_2k_3) \geq \wedge\{I_{\mathcal{P}}(k_2), \varepsilon_I\} \\ F_{\mathcal{P}}(k_1k_2k_3) \leq \vee\{F_{\mathcal{P}}(k_2), \varepsilon_F\} \end{pmatrix}$.

(ii) $(\forall k_2, k_3 \in \mathbb{N}),\ k_2 \leq k_3 \Rightarrow \begin{pmatrix} T_{\mathcal{P}}(k_2) \leq T_{\mathcal{P}}(k_3) \vee \varepsilon_T \\ I_{\mathcal{P}}(k_2) \geq I_{\mathcal{P}}(k_3) \wedge \varepsilon_I \\ F_{\mathcal{P}}(k_2) \leq F_{\mathcal{P}}(k_3) \vee \varepsilon_F \end{pmatrix}$

where $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [-1, 0]$ such that $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$.

Theorem 3.14. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. If $\mathbb{N}_{\mathcal{K}}$ and $\mathbb{N}_{\mathcal{P}}$ are an ε-neutrosophic $\varkappa$-subsemigroup and a δ-neutrosophic $\varkappa$-subsemigroup, respectively in $\mathbb{N}$ for any $\varepsilon_T, \varepsilon_I, \varepsilon_F, \delta_T, \delta_I, \delta_F \in [-1, 0]$ with $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$ and $-3 \leq \delta_T + \delta_I + \delta_F \leq 0$, then $\mathbb{N}_{\mathcal{K}} \cap \mathbb{N}_{\mathcal{P}}$ is a ν-neutrosophic $\varkappa$-subsemigroup of $\mathbb{N}$ for $\nu := \varepsilon \wedge \delta$, that is, $(\nu_T, \nu_I, \nu_F) = (\varepsilon_T \vee \delta_T, \varepsilon_I \wedge \delta_I, \varepsilon_F \vee \delta_F)$.

Proof. The proof is similar to Theorem 4.14 of [14].

Theorem 3.15. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. If $\mathbb{N}_{\mathcal{K}}$ and $\mathbb{N}_{\mathcal{P}}$ are an ε-neutrosophic $\varkappa$-interior ideal and a δ-neutrosophic $\varkappa$-interior ideal, respectively in $\mathbb{N}$ for any $\varepsilon_T, \varepsilon_I, \varepsilon_F, \delta_T, \delta_I, \delta_F \in [-1, 0]$ with $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$ and $-3 \leq \delta_T + \delta_I + \delta_F \leq 0$, then $\mathbb{N}_{\mathcal{K}} \cap \mathbb{N}_{\mathcal{P}}$ is a ν-neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$ for $\nu := \varepsilon \wedge \delta$, that is, $(\nu_T, \nu_I, \nu_F) = (\varepsilon_T \vee \delta_T, \varepsilon_I \wedge \delta_I, \varepsilon_F \vee \delta_F)$.

Proof. For any $k_1, k_2, k_3 \in \mathbb{N}$, we have

$$\begin{aligned}
T_{\mathcal{K}\cap\mathcal{P}}(k_1k_2k_3) &= \vee\{T_{\mathcal{K}}(k_1k_2k_3), T_{\mathcal{P}}(k_1k_2k_3)\} \\
&\leq \vee\{\vee\{T_{\mathcal{K}}(k_2), \varepsilon_T\}, \vee\{T_{\mathcal{P}}(k_2), \delta_T\}\} \\
&\leq \vee\{\vee\{T_{\mathcal{K}}(k_2), \nu_T\}, \vee\{T_{\mathcal{P}}(k_2), \nu_T\}\} \\
&= \vee\{T_{\mathcal{K}}(k_2), T_{\mathcal{P}}(k_2), \nu_T\} \\
&= \vee\{T_{\mathcal{K}\cap\mathcal{P}}(k_2), \nu_T\}, \\
I_{\mathcal{K}\cap\mathcal{P}}(k_1k_2k_3) &= \wedge\{I_{\mathcal{K}}(k_1k_2k_3), I_{\mathcal{P}}(k_1k_2k_3)\} \\
&\geq \wedge\{\wedge\{I_{\mathcal{K}}(k_2), \varepsilon_I\}, \wedge\{I_{\mathcal{P}}(k_2), \delta_I\}\} \\
&\geq \wedge\{\wedge\{I_{\mathcal{K}}(k_2), \nu_I\}, \wedge\{I_{\mathcal{P}}(k_2), \nu_I\}\} \\
&= \wedge\{I_{\mathcal{K}}(k_2), I_{\mathcal{P}}(k_2), \nu_I\} \\
&= \wedge\{I_{\mathcal{K}\cap\mathcal{P}}(k_2), \nu_I\},
\end{aligned}$$

$$\begin{aligned}F_{\mathcal{K}\cap\mathcal{P}}(k_1k_2k_3) &= \vee\{F_{\mathcal{K}}(k_1k_2k_3), F_{\mathcal{P}}(k_1k_2k_3)\}\\ &\leq \vee\{\vee\{F_{\mathcal{K}}(k_2),\varepsilon_F\},\vee\{F_{\mathcal{P}}(k_2),\delta_F\}\}\\ &\leq \vee\{\vee\{F_{\mathcal{K}}(k_2),\nu_F\},\vee\{F_{\mathcal{P}}(k_2),\nu_F\}\}\\ &= \vee\{F_{\mathcal{K}}(k_2),F_{\mathcal{P}}(k_2),\nu_F\}\\ &= \vee\{F_{\mathcal{K}\cap\mathcal{P}}(k_2),\nu_F\}.\end{aligned}$$

For any $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq k_2$, we have

$$\begin{aligned}T_{\mathcal{K}\cap\mathcal{P}}(k_1) &= T_{\mathcal{K}}(k_1)\vee T_{\mathcal{P}}(k_1)\\ &\leq \{T_{\mathcal{K}}(k_2)\vee\varepsilon_T\}\vee\{T_{\mathcal{P}}(k_2)\vee\delta_T\}\\ &= T_{\mathcal{K}\cap\mathcal{P}}(k_2)\vee\nu_T,\\ I_{\mathcal{K}\cap\mathcal{P}}(k_1) &= I_{\mathcal{K}}(k_1)\wedge I_{\mathcal{P}}(k_1)\\ &\geq \{I_{\mathcal{K}}(k_2)\wedge\varepsilon_T\}\wedge\{I_{\mathcal{P}}(k_2)\wedge\delta_T\}\\ &= I_{\mathcal{K}\cap\mathcal{P}}(k_2)\wedge\nu_T,\\ F_{\mathcal{K}\cap\mathcal{P}}(k_1) &= F_{\mathcal{K}}(k_1)\vee F_{\mathcal{P}}(k_1)\\ &\leq \{F_{\mathcal{K}}(k_2)\vee\varepsilon_T\}\vee\{F_{\mathcal{P}}(k_2)\vee\delta_T\}\\ &= F_{\mathcal{K}\cap\mathcal{P}}(k_2)\vee\nu_T.\end{aligned}$$

Therefore $\mathbb{N}_{\mathcal{K}}\cap\mathbb{N}_{\mathcal{P}}$ is a ν-neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$.

Theorem 3.16. Let $(\mathbb{N}, ., \leq)$ be an ordered semigroup. Let $\mathbb{N}_{\mathcal{K}}$ be an ε-neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$. If

$$\kappa := (\kappa_T,\kappa_I,\kappa_F) = \left(\bigvee_{k_1\in\mathbb{N}} T_{\mathcal{K}}(k_1),\ \bigwedge_{k_1\in\mathbb{N}} I_{\mathcal{K}}(k_1),\ \bigvee_{k_1\in\mathbb{N}} F_{\mathcal{K}}(k_1)\right),$$

then the set

$$\Omega := \{k\in\mathbb{N} \mid T_{\mathcal{K}}(k)\leq\kappa_T\vee\varepsilon_T,\ I_{\mathcal{K}}(k)\geq\kappa_I\wedge\varepsilon_I,\ F_{\mathcal{K}}(k)\leq\kappa_F\vee\varepsilon_F\}$$

is an interior ideal of $\mathbb{N}$.

Proof. Let $k_1, k_2 \in \mathbb{N}$. If $k_1, k_2 \in \Omega$, then

$$\begin{aligned}&T_{\mathcal{K}}(k_1)\leq\kappa_T\vee\varepsilon_T = \bigvee_{k_1\in\mathbb{N}}\{T_{\mathcal{K}}(k_1)\}\vee\varepsilon_T, \quad T_{\mathcal{K}}(k_2)\leq\kappa_T\vee\varepsilon_T = \bigvee_{k_2\in\mathbb{N}}\{T_{\mathcal{K}}(k_2)\}\vee\varepsilon_T,\\ &I_{\mathcal{K}}(k_1)\geq\kappa_I\wedge\varepsilon_I = \bigwedge_{k_1\in\mathbb{N}}\{I_{\mathcal{K}}(k_1)\}\wedge\varepsilon_I, \quad I_{\mathcal{K}}(k_2)\geq\kappa_I\wedge\varepsilon_I = \bigwedge_{k_2\in\mathbb{N}}\{I_{\mathcal{K}}(k_2)\}\wedge\varepsilon_I,\\ &F_{\mathcal{K}}(k_1)\leq\kappa_F\vee\varepsilon_F = \bigvee_{k_1\in\mathbb{N}}\{F_{\mathcal{K}}(k_1)\}\vee\varepsilon_F, \quad F_{\mathcal{K}}(k_2)\leq\kappa_F\vee\varepsilon_F = \bigvee_{k_2\in\mathbb{N}}\{F_{\mathcal{K}}(k_2)\}\vee\varepsilon_F.\end{aligned}$$

Now for any $k_1, k_2, k_3 \in \mathbb{N}$,

$$T_{\mathcal{K}}(k_1k_2k_3)\leq\bigvee\{T_{\mathcal{K}}(k_2),\varepsilon_T\}\leq\bigvee\{\kappa_T\vee\varepsilon_T,\varepsilon_T\} = \kappa_T\vee\varepsilon_T,$$

$$I_{\mathcal{K}}(k_1k_2k_3)\geq\bigwedge\{I_{\mathcal{K}}(k_2),\varepsilon_I\}\geq\bigwedge\{\kappa_I\wedge\varepsilon_I,\varepsilon_I\} = \kappa_I\wedge\varepsilon_I,$$

$$F_{\mathcal{K}}(k_1k_2k_3) \leq \bigvee\{F_{\mathcal{K}}(k_2), \varepsilon_F\} \leq \bigvee\{\kappa_F \vee \varepsilon_F, \varepsilon_F\} = \kappa_F \vee \varepsilon_F.$$

So $k_1k_2k_3 \in \Omega$.

Let $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq k_2$ and $k_2 \in \Omega$. Then $T_{\mathcal{K}}(k_2) \leq \kappa_T \vee \varepsilon_T$, $I_{\mathcal{K}}(k_2) \geq \kappa_I \wedge \varepsilon_I$, $F_{\mathcal{K}}(k_2) \leq \kappa_F \vee \varepsilon_F$. Since $\mathbb{N}_{\mathcal{K}}$ is an ε-neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$, we have $T_{\mathcal{K}}(k_1) \leq T_{\mathcal{K}}(k_2)$, $I_{\mathcal{K}}(k_1) \geq I_{\mathcal{K}}(k_2)$ and $F_{\mathcal{K}}(k_1) \leq F_{\mathcal{K}}(k_2)$. So $T_{\mathcal{K}}(k_1) \leq \kappa_T \vee \varepsilon_T$, $I_{\mathcal{K}}(k_1) \geq \kappa_I \wedge \varepsilon_I$, $F_{\mathcal{K}}(k_1) \leq \kappa_F \vee \varepsilon_F$ which imply $k_1 \in \Omega$ and hence Ω is an interior ideal of $\mathbb{N}$.

Following [26], let $(\mathbb{N}, ., \leq)$ and $(\mathbb{M}, *, \preceq)$ be ordered semigroups and a mapping $f:\mathbb{N} \to \mathbb{M}$. f is known as isotone if $k_1, k_2 \in \mathbb{N}$, $k_1 \leq k_2$ implies $f(k_1) \preceq f(k_2)$. f is called inverse isotone if $k_1, k_2 \in \mathbb{N}$, $f(k_1) \preceq f(k_2)$ implies $k_1 \leq k_2$ [each inverse isotone mapping is (1–1)]. f is said to be a homomorphism if (i) f is isotone and (ii) $f(k_1.k_2) = f(k_1) * f(k_2)$ for all $k_1, k_2 \in \mathbb{N}$. f is known as an isomorphism if it is homomorphism, onto and inverse isotone.

For a map $f : \mathbb{N} \to \mathbb{M}$ of ordered semigroups and a neutrosophic $\varkappa$-structure $\mathbb{N}_{\mathcal{K}} := \frac{\mathbb{M}}{(T_{\mathcal{K}}, I_{\mathcal{K}}, F_{\mathcal{K}})}$ over $\mathbb{M}$ and $\varepsilon = (\varepsilon_T, \varepsilon_I, \varepsilon_F)$ with $-3 \leq \varepsilon_T + \varepsilon_I + \varepsilon_F \leq 0$, define a neutrosophic $\varkappa$-structure $\mathbb{N}^{\varepsilon}_{\mathcal{K}} := \frac{\mathbb{N}}{(T^{\varepsilon}_{\mathcal{K}}, I^{\varepsilon}_{\mathcal{K}}, F^{\varepsilon}_{\mathcal{K}})}$ over $\mathbb{N}$ by:

$$T^{\varepsilon}_{\mathcal{K}} : \mathbb{N} \to [-1, 0],\ k \mapsto \bigvee\{T_{\mathcal{K}}(f(k)), \varepsilon_T\},$$
$$I^{\varepsilon}_{\mathcal{K}} : \mathbb{N} \to [-1, 0],\ k \mapsto \bigwedge\{I_{\mathcal{K}}(f(k)), \varepsilon_I\},$$
$$F^{\varepsilon}_{\mathcal{K}} : \mathbb{N} \to [-1, 0],\ k \mapsto \bigvee\{F_{\mathcal{K}}(f(k)), \varepsilon_F\}.$$

Theorem 3.17. Let $(\mathbb{N}, ., \leq)$ and $(\mathbb{M}, *, \preceq)$ be ordered semigroups and $f : \mathbb{N} \to \mathbb{M}$ be homomorphism. If a neutrosophic $\varkappa$-structure $\mathbb{N}_{\mathcal{K}} := \frac{\mathbb{M}}{(T_{\mathcal{K}}, I_{\mathcal{K}}, F_{\mathcal{K}})}$ over $\mathbb{M}$ is an ε-neutrosophic $\varkappa$-subsemigroup of $\mathbb{M}$, then $\mathbb{N}^{\varepsilon}_{\mathcal{K}} := \frac{\mathbb{N}}{(T^{\varepsilon}_{\mathcal{K}}, I^{\varepsilon}_{\mathcal{K}}, F^{\varepsilon}_{\mathcal{K}})}$ is an ε-neutrosophic $\varkappa$-subsemigroup in $\mathbb{N}$.

Proof. The proof is similar to Theorem 4.16 of [14].

Theorem 3.18. Let $(\mathbb{N}, ., \leq)$ and $(\mathbb{M}, *, \preceq)$ be ordered semigroups and $f : \mathbb{N} \to \mathbb{M}$ be homomorphism. If a neutrosophic $\varkappa$-structure $\mathbb{N}_{\mathcal{K}} := \frac{\mathbb{M}}{(T_{\mathcal{K}}, I_{\mathcal{K}}, F_{\mathcal{K}})}$ over $\mathbb{M}$ is an ε-neutrosophic $\varkappa$-left ideal (resp., right ideal, ideal) of $\mathbb{M}$, then $\mathbb{N}^{\varepsilon}_{\mathcal{K}} := \frac{\mathbb{N}}{(T^{\varepsilon}_{\mathcal{K}}, I^{\varepsilon}_{\mathcal{K}}, F^{\varepsilon}_{\mathcal{K}})}$ is an ε-neutrosophic $\varkappa$-left ideal (resp., right ideal, ideal) of $\mathbb{N}$.

Proof. Let $\mathbb{N}_{\mathcal{K}} := \frac{\mathbb{M}}{(T_{\mathcal{K}}, I_{\mathcal{K}}, F_{\mathcal{K}})}$ is an ε-neutrosophic $\varkappa$-left ideal in $\mathbb{M}$. For any $k_1, k_2 \in \mathbb{N}$, we have

$$T^{\varepsilon}_{\mathcal{K}}(k_1k_2) = \bigvee\{T_{\mathcal{K}}(f(k_1k_2)), \varepsilon_T\} = \bigvee\{T_{\mathcal{K}}(f(k_1) * f(k_2)), \varepsilon_T\} \preceq \bigvee\{T_{\mathcal{K}}(f(k_2)), \varepsilon_T\} = T^{\varepsilon}_{\mathcal{K}}(k_2),$$

$$I^{\varepsilon}_{\mathcal{K}}(k_1k_2) = \bigwedge\{I_{\mathcal{K}}(f(k_1k_2)), \varepsilon_I\} = \bigwedge\{I_{\mathcal{K}}(f(k_1) * f(k_2)), \varepsilon_I\} \succeq \bigwedge\{I_{\mathcal{K}}(f(k_2)), \varepsilon_I\} = I^{\varepsilon}_{\mathcal{K}}(k_2),$$

$$F^{\varepsilon}_{\mathcal{K}}(k_1k_2) = \bigvee\{F_{\mathcal{K}}(f(k_1k_2)), \varepsilon_F\} = \bigvee\{F_{\mathcal{K}}(f(k_1) * f(k_2)), \varepsilon_F\} \preceq \bigvee\{F_{\mathcal{K}}(f(k_2)), \varepsilon_F\} = F^{\varepsilon}_{\mathcal{K}}(k_2).$$

Let $k_1 \leq k_2$. Then

$$T^{\varepsilon}_{\mathcal{K}}(k_1) = T_{\mathcal{K}}(f(k_1)) \vee \varepsilon_T \preceq T_{\mathcal{K}}(f(k_2)) \vee \varepsilon_T = T^{\varepsilon}_{\mathcal{K}}(k_2),$$
$$I^{\varepsilon}_{\mathcal{K}}(k_1) = I_{\mathcal{K}}(f(k_1)) \wedge \varepsilon_I \succeq I_{\mathcal{K}}(f(k_2)) \wedge \varepsilon_I = I^{\varepsilon}_{\mathcal{K}}(k_2),$$
$$F^{\varepsilon}_{\mathcal{K}}(k_1) = F_{\mathcal{K}}(f(k_1)) \vee \varepsilon_F \preceq F_{\mathcal{K}}(f(k_2)) \vee \varepsilon_F = F^{\varepsilon}_{\mathcal{K}}(k_2).$$

Therefore $\mathbb{N}^{\varepsilon}_{\mathcal{K}}$ is an ε-neutrosophic $\varkappa$-left ideal in $\mathbb{N}$.

Theorem 3.19. Let $(\mathbb{N}, ., \leq)$ and $(\mathbb{M}, *, \preceq)$ be ordered semigroups and $f : \mathbb{N} \to \mathbb{M}$ be homomorphism. If a neutrosophic $\varkappa$-structure $\mathbb{N}_{\mathcal{K}} := \frac{\mathbb{M}}{(T_{\mathcal{K}}, I_{\mathcal{K}}, F_{\mathcal{K}})}$ over $\mathbb{M}$ is an ε-neutrosophic $\varkappa$-interior ideal of $\mathbb{M}$, then $\mathbb{N}^{\varepsilon}_{\mathcal{K}} := \frac{\mathbb{N}}{(T^{\varepsilon}_{\mathcal{K}}, I^{\varepsilon}_{\mathcal{K}}, F^{\varepsilon}_{\mathcal{K}})}$ is an ε-neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$.

Proof. For any $k_1, k_2, k_3 \in \mathbb{N}$, we have

$$
\begin{aligned}
T^{\varepsilon}_{\mathcal{K}}(k_1k_2k_3) &= \bigvee\{T_{\mathcal{K}}(f(k_1k_2k_3)), \varepsilon_T\}\\
&= \bigvee\{T_{\mathcal{K}}(f(k_1) * f(k_2) * f(k_3)), \varepsilon_T\}\\
&\preceq \bigvee\{T_{\mathcal{K}}(f(k_2)), \varepsilon_T\}\\
&= T^{\varepsilon}_{\mathcal{K}}(k_2),\\
I^{\varepsilon}_{\mathcal{K}}(k_1k_2k_3) &= \bigwedge\{I_{\mathcal{K}}(f(k_1k_2k_3)), \varepsilon_I\}\\
&= \bigwedge\{I_{\mathcal{K}}(f(k_1) * f(k_2) * f(k_3)), \varepsilon_I\}\\
&\succeq \bigwedge\{I_{\mathcal{K}}(f(k_2)), \varepsilon_I\}\\
&= I^{\varepsilon}_{\mathcal{K}}(k_2),\\
F^{\varepsilon}_{\mathcal{K}}(k_1k_2k_3) &= \bigvee\{F_{\mathcal{K}}(f(k_1k_2k_3)), \varepsilon_F\}\\
&= \bigvee\{F_{\mathcal{K}}(f(k_1) * f(k_2) * f(k_3)), \varepsilon_F\}\\
&\preceq \bigvee\{F_{\mathcal{K}}(f(k_2)), \varepsilon_F\}\\
&= F^{\varepsilon}_{\mathcal{K}}(k_2).
\end{aligned}
$$

Let $k_1 \leq k_2$. Then

$$
\begin{aligned}
T^{\varepsilon}_{\mathcal{K}}(k_1) &= T_{\mathcal{K}}(f(k_1)) \vee \varepsilon_T \preceq T_{\mathcal{K}}(f(k_2)) \vee \varepsilon_T = T^{\varepsilon}_{\mathcal{K}}(k_2),\\
I^{\varepsilon}_{\mathcal{K}}(k_1) &= I_{\mathcal{K}}(f(k_1)) \wedge \varepsilon_I \succeq I_{\mathcal{K}}(f(k_2)) \wedge \varepsilon_I = I^{\varepsilon}_{\mathcal{K}}(k_2),\\
F^{\varepsilon}_{\mathcal{K}}(k_1) &= F_{\mathcal{K}}(f(k_1)) \vee \varepsilon_F \preceq F_{\mathcal{K}}(f(k_2)) \vee \varepsilon_F = F^{\varepsilon}_{\mathcal{K}}(k_2).
\end{aligned}
$$

Hence by Theorem 3.17, $\mathbb{N}^{\varepsilon}_{\mathcal{K}}$ is an ε-neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$.

Let $(\mathbb{N}, ., \leq)$ and $(\mathbb{M}, *, \preceq)$ be ordered semigroups. Consider a map $f : \mathbb{N} \to \mathbb{M}$. If $\mathbb{M}_{\mathcal{P}} := \frac{\mathbb{M}}{(T_{\mathcal{P}}, I_{\mathcal{P}}, F_{\mathcal{P}})}$ is a neutrosophic $\varkappa$-structures over $\mathbb{M}$, then the preimage of $\mathbb{M}_{\mathcal{P}}$ under f is defined to be a neutrosophic $\varkappa$-structures

$$
f^{-1}(\mathbb{M}_{\mathcal{P}}) = \frac{\mathbb{N}}{(f^{-1}(T_{\mathcal{P}}), f^{-1}(I_{\mathcal{P}}), f^{-1}(F_{\mathcal{P}}))}
$$

over $\mathbb{N}$, where $f^{-1}(T_{\mathcal{P}})(k) = T_{\mathcal{P}}(f(k))$, $f^{-1}(I_{\mathcal{P}})(k) = I_{\mathcal{P}}(f(k))$ and $f^{-1}(F_{\mathcal{P}})(k) = F_{\mathcal{P}}(f(k))$ for all $k \in \mathbb{N}$.

Theorem 3.20. Let $(\mathbb{N}, ., \leq)$ and $(\mathbb{M}, *, \preceq)$ be ordered semigroups and $f: \mathbb{N} \to \mathbb{M}$ be homomorphism. If $\mathbb{M}_{\mathcal{K}} := \frac{\mathbb{M}}{(T_{\mathcal{P}}, I_{\mathcal{P}}, F_{\mathcal{P}})}$ is a neutrosophic $\varkappa$-subsemigroup of $\mathbb{M}$, then the preimage of $\mathbb{M}_{\mathcal{K}}$ under f is a neutrosophic $\varkappa$-subsemigroup of $\mathbb{N}$.

Proof. The proof is similar to Theorem 4.17 of [14].

Theorem 3.21. Let $(\mathbb{N}, ., \leq)$ and $(\mathbb{M}, *, \preceq)$ be ordered semigroups and $f: \mathbb{N} \to \mathbb{M}$ be homomorphism. If $\mathbb{M}_{\mathcal{P}} := \frac{\mathbb{M}}{(T_{\mathcal{P}}, I_{\mathcal{P}}, F_{\mathcal{P}})}$ is a neutrosophic $\varkappa$-right ideal (resp., left ideal, ideal) of $\mathbb{M}$, then the preimage of $\mathbb{M}_{\mathcal{P}}$ under f is a neutrosophic $\varkappa$-right ideal (resp., left ideal, ideal) of $\mathbb{N}$.

Proof. Let $f^{-1}(\mathbb{M}_{\mathcal{P}}) = \frac{\mathbb{N}}{(f^{-1}(T_{\mathcal{P}}), f^{-1}(I_{\mathcal{P}}), f^{-1}(F_{\mathcal{P}}))}$ be the preimage of $\mathbb{M}_{\mathcal{P}}$ under the map f. For any $k_1, k_2 \in \mathbb{N}$, we can have

$$f^{-1}(T_{\mathcal{P}})(k_1k_2) = T_{\mathcal{P}}(f(k_1k_2)) = T_{\mathcal{P}}(f(k_1) * f(k_2)) \preceq T_{\mathcal{P}}(f(k_1)) = f^{-1}(T_{\mathcal{P}})(k_1),$$

$$f^{-1}(I_{\mathcal{P}})(k_1k_2) = I_{\mathcal{P}}(f(k_1k_2)) = I_{\mathcal{P}}(f(k_1) * f(k_2)) \succeq I_{\mathcal{P}}(f(k_1)) = f^{-1}(I_{\mathcal{P}})(k_1),$$

$$f^{-1}(F_{\mathcal{P}})(k_1k_2) = F_{\mathcal{P}}(f(k_1k_2)) = F_{\mathcal{P}}(f(k_1) * f(k_2)) \preceq F_{\mathcal{P}}(f(k_1)) = f^{-1}(F_{\mathcal{P}})(k_1).$$

Let $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq k_2$. Then

$$f^{-1}(T_{\mathcal{P}})(k_1) = T_{\mathcal{P}}(f(k_1)) \preceq T_{\mathcal{P}}(f(k_2)) = f^{-1}(T_{\mathcal{P}})(k_2),$$

$$f^{-1}(I_{\mathcal{P}})(k_1) = I_{\mathcal{P}}(f(k_1)) \succeq I_{\mathcal{P}}(f(k_2)) = f^{-1}(I_{\mathcal{P}})(k_2),$$

$$f^{-1}(F_{\mathcal{P}})(k_1) = F_{\mathcal{P}}(f(k_1)) \preceq F_{\mathcal{P}}(f(k_2)) = f^{-1}(F_{\mathcal{P}})(k_2).$$

Therefore $f^{-1}(\mathbb{M}_{\mathcal{P}})$ is a neutrosophic $\varkappa$-right ideal in $\mathbb{N}$.

Theorem 3.22. Let $(\mathbb{N}, ., \leq)$ and $(\mathbb{M}, *, \preceq)$ be ordered semigroups. Consider the homomorphism $f: \mathbb{N} \to \mathbb{M}$. If $\mathbb{M}_{\mathcal{P}} := \frac{\mathbb{M}}{(T_{\mathcal{P}}, I_{\mathcal{P}}, F_{\mathcal{P}})}$ is a neutrosophic $\varkappa$-interior ideal of $\mathbb{M}$, then the preimage of $\mathbb{M}_{\mathcal{P}}$ under f is a neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$.

Proof. Let $f^{-1}(\mathbb{M}_{\mathcal{P}}) = \frac{\mathbb{N}}{(f^{-1}(T_{\mathcal{P}}), f^{-1}(I_{\mathcal{P}}), f^{-1}(F_{\mathcal{P}}))}$ be the preimage of $\mathbb{M}_{\mathcal{P}}$ under f. For any $k_1, k_2, k_3 \in \mathbb{N}$, we have

$$f^{-1}(T_{\mathcal{P}})(k_1k_2k_3) = T_{\mathcal{P}}(f(k_1k_2k_3)) = T_{\mathcal{P}}(f(k_1) * f(k_2) * f(k_3)) \preceq T_{\mathcal{P}}(f(k_2)) = f^{-1}(T_{\mathcal{P}})(k_2),$$

$$f^{-1}(I_{\mathcal{P}})(k_1k_2k_3) = I_{\mathcal{P}}(f(k_1k_2k_3)) = I_{\mathcal{P}}(f(k_1) * f(k_2) * f(k_3)) \succeq I_{\mathcal{P}}(f(k_2)) = f^{-1}(I_{\mathcal{P}})(k_2),$$

$$f^{-1}(F_{\mathcal{P}})(k_1k_2k_3) = F_{\mathcal{P}}(f(k_1k_2k_3)) = F_{\mathcal{P}}(f(k_1) * f(k_2) * f(k_3)) \preceq F_{\mathcal{P}}(f(k_2)) = f^{-1}(F_{\mathcal{P}})(k_2).$$

Let $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq k_2$. Then

$$f^{-1}(T_{\mathcal{P}})(k_1) = T_{\mathcal{P}}(f(k_1)) \preceq T_{\mathcal{P}}(f(k_2)) = f^{-1}(T_{\mathcal{P}})(k_2),$$

$$f^{-1}(I_{\mathcal{P}})(k_1) = I_{\mathcal{P}}(f(k_1)) \succeq I_{\mathcal{P}}(f(k_2)) = f^{-1}(I_{\mathcal{P}})(k_2),$$

$$f^{-1}(F_{\mathcal{P}})(k_1) = F_{\mathcal{P}}(f(k_1)) \preceq F_{\mathcal{P}}(f(k_2)) = f^{-1}(F_{\mathcal{P}})(k_2).$$

Therefore, by Theorem 3.20, $f^{-1}(\mathbb{M}_{\mathcal{P}})$ is a neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$.

Let $(\mathbb{N}, ., \leq)$ and $(\mathbb{M}, *, \preceq)$ be ordered semigroups. Consider the onto function $f : \mathbb{N} \to \mathbb{M}$. If $\mathbb{N}_{\mathcal{K}} := \frac{\mathbb{N}}{(T_{\mathcal{K}}, I_{\mathcal{K}}, F_{\mathcal{K}})}$ is a neutrosophic $\varkappa$-structures over $\mathbb{N}$, then the image of $\mathbb{N}_{\mathcal{K}}$ under f is defined to be a neutrosophic $\varkappa$-structures

$$f(\mathbb{N}_{\mathcal{K}}) = \frac{\mathbb{M}}{(f(T_{\mathcal{K}}), f(I_{\mathcal{K}}), f(F_{\mathcal{K}}))}$$

over $\mathbb{M}$, where for all $k_2 \in \mathbb{M}$,

$$f(T_{\mathcal{K}})(k_2) = \bigwedge_{k_1 \in f^{-1}(k_2)} T_{\mathcal{K}}(k_1),$$

$$f(I_{\mathcal{K}})(k_2) = \bigvee_{k_1 \in f^{-1}(k_2)} I_{\mathcal{K}}(k_1),$$

$$f(F_{\mathcal{K}})(k_2) = \bigwedge_{k_1 \in f^{-1}(k_2)} F_{\mathcal{K}}(k_1).$$

Theorem 3.23. Let $(\mathbb{N}, ., \leq)$ and $(\mathbb{M}, *, \preceq)$ be ordered semigroups. Consider an onto homomorphism $f : \mathbb{N} \to \mathbb{M}$. Let $\mathbb{N}_{\mathcal{K}} := \frac{\mathbb{N}}{(T_{\mathcal{K}}, I_{\mathcal{K}}, F_{\mathcal{K}})}$ be a neutrosophic $\varkappa$-structure over $\mathbb{N}$ such that

$$(\forall Q \subseteq \mathbb{N})\ \ (\exists x_0 \in Q) \begin{pmatrix} T_{\mathcal{K}}(x_0) = \bigwedge_{z \in Q} T_{\mathcal{K}}(z) \\ I_{\mathcal{K}}(x_0) = \bigvee_{z \in Q} I_{\mathcal{K}}(z) \\ F_{\mathcal{K}}(x_0) = \bigwedge_{z \in Q} F_{\mathcal{K}}(z) \end{pmatrix}.$$

If $\mathbb{N}_{\mathcal{K}}$ is a neutrosophic $\varkappa$-subsemigroup of $\mathbb{N}$, then the image of $\mathbb{N}_{\mathcal{K}}$ is a neutrosophic $\varkappa$-subsemigroup of $\mathbb{M}$ under f.

Proof. The proof is similar to Theorem 4.18 of [14].

Theorem 3.24. Let $(\mathbb{N}, ., \leq)$ and $(\mathbb{M}, *, \preceq)$ be ordered semigroups. Consider an onto homomorphism $f : \mathbb{N} \to \mathbb{M}$. Let $\mathbb{N}_{\mathcal{K}} := \frac{\mathbb{N}}{(T_{\mathcal{K}}, I_{\mathcal{K}}, F_{\mathcal{K}})}$ be a neutrosophic $\varkappa$-structure over $\mathbb{N}$ such that

$$(\forall Q \subseteq \mathbb{N})\ \ (\exists x_0 \in Q) \begin{pmatrix} T_{\mathcal{K}}(x_0) = \bigwedge_{z \in Q} T_{\mathcal{K}}(z) \\ I_{\mathcal{K}}(x_0) = \bigvee_{z \in Q} I_{\mathcal{K}}(z) \\ F_{\mathcal{K}}(x_0) = \bigwedge_{z \in Q} F_{\mathcal{K}}(z) \end{pmatrix}.$$

If $\mathbb{N}_{\mathcal{K}}$ is a neutrosophic $\varkappa$-left ideal in $\mathbb{N}$, then the image of $\mathbb{N}_{\mathcal{K}}$ is a neutrosophic $\varkappa$-left ideal in $\mathbb{M}$ under f.

Proof. Let $f(\mathbb{N}_{\mathcal{K}}) = \frac{\mathbb{M}}{(f(T_{\mathcal{K}}), f(I_{\mathcal{K}}), f(F_{\mathcal{K}}))}$ be the image in $\mathbb{N}_{\mathcal{K}}$ under f and $k_1, k_2 \in \mathbb{M}$. Then $f^{-1}(k_1) \neq \phi$ and $f^{-1}(k_2) \neq \phi$. So there exist $s \in f^{-1}(k_1)$ and $t \in f^{-1}(k_2)$ such that

$$T_{\mathcal{K}}(s) = \bigwedge_{u \in f^{-1}(k_1)} T_{\mathcal{K}}(u), \quad T_{\mathcal{K}}(t) = \bigwedge_{v \in f^{-1}(k_2)} T_{\mathcal{K}}(v),$$
$$I_{\mathcal{K}}(s) = \bigvee_{u \in f^{-1}(k_1)} I_{\mathcal{K}}(u), \quad I_{\mathcal{K}}(t) = \bigvee_{v \in f^{-1}(k_2)} I_{\mathcal{K}}(v),$$
$$F_{\mathcal{K}}(s) = \bigwedge_{u \in f^{-1}(k_1)} F_{\mathcal{K}}(u), \quad F_{\mathcal{K}}(t) = \bigwedge_{v \in f^{-1}(k_2)} F_{\mathcal{K}}(v).$$

Now,

$$\begin{aligned}
f(T_{\mathcal{K}})(k_1k_2) &= \bigwedge_{x \in f^{-1}(k_1k_2)} T_{\mathcal{K}}(x) \\
&\leq T_{\mathcal{K}}(st) \\
&\leq T_{\mathcal{K}}(t) \\
&= \bigwedge_{v \in f^{-1}(k_2)} T_{\mathcal{K}}(v) \\
&= f(T_{\mathcal{K}})(k_2),
\end{aligned}$$

$$\begin{aligned}
f(I_{\mathcal{K}})(k_1k_2) &= \bigvee_{x \in f^{-1}(k_1k_2)} I_{\mathcal{K}}(x) \\
&\geq I_{\mathcal{K}}(st) \\
&\geq I_{\mathcal{K}}(t) \\
&= \bigvee_{v \in f^{-1}(k_2)} I_{\mathcal{K}}(v) \\
&= f(I_{\mathcal{K}})(k_2),
\end{aligned}$$

$$\begin{aligned}
f(F_{\mathcal{K}})(k_1k_2) &= \bigwedge_{x \in f^{-1}(k_1k_2)} F_{\mathcal{K}}(x) \\
&\leq F_{\mathcal{K}}(st) \\
&\leq F_{\mathcal{K}}(t) \\
&= \bigwedge_{v \in f^{-1}(k_2)} F_{\mathcal{K}}(v) \\
&= f(F_{\mathcal{K}})(k_2).
\end{aligned}$$

Let $k_1, k_2 \in M$ with $k_1 \leq k_2$. Then

$$\begin{aligned}
f(T_{\mathcal{K}})(k_1) &= \bigwedge_{x \in f^{-1}(k_1)} T_{\mathcal{K}}(x) \\
&= T_{\mathcal{K}}(s) \\
&\leq T_{\mathcal{K}}(t) \\
&= \bigwedge_{v \in f^{-1}(k_2)} T_{\mathcal{K}}(v) \\
&= f(T_{\mathcal{K}})(k_2), \\
f(I_{\mathcal{K}})(k_1) &= \bigvee_{x \in f^{-1}(k_1)} I_{\mathcal{K}}(x) \\
&= I_{\mathcal{K}}(s) \\
&\geq I_{\mathcal{K}}(t) \\
&= \bigvee_{v \in f^{-1}(k_2)} I_{\mathcal{K}}(v) \\
&= f(I_{\mathcal{K}})(k_2), \\
f(F_{\mathcal{K}})(k_1) &= \bigwedge_{x \in f^{-1}(k_1)} F_{\mathcal{K}}(x) \\
&= F_{\mathcal{K}}(s) \\
&\leq F_{\mathcal{K}}(t) \\
&= \bigwedge_{v \in f^{-1}(k_2)} F_{\mathcal{K}}(v) \\
&= f(F_{\mathcal{K}})(k_2).
\end{aligned}$$

Therefore $f(\mathbb{N}_{\mathcal{K}})$ is a neutrosophic $\varkappa$-left ideal in $\mathbb{M}$.

Theorem 3.25. Let $(\mathbb{N}, ., \leq)$ and $(\mathbb{M}, *, \preceq)$ be ordered semigroups. Consider an onto homomorphism $f : \mathbb{N} \to \mathbb{M}$. Let $\mathbb{N}_{\mathcal{K}} := \frac{\mathbb{N}}{(T_{\mathcal{K}}, I_{\mathcal{K}}, F_{\mathcal{K}})}$ be a neutrosophic $\varkappa$-structure over $\mathbb{N}$ such that

$$(\forall Q \subseteq \mathbb{N})\ (\exists x_0 \in Q) \begin{pmatrix} T_{\mathcal{K}}(x_0) = \bigwedge_{z \in Q} T_{\mathcal{K}}(z) \\ I_{\mathcal{K}}(x_0) = \bigvee_{z \in Q} I_{\mathcal{K}}(z) \\ F_{\mathcal{K}}(x_0) = \bigwedge_{z \in Q} F_{\mathcal{K}}(z) \end{pmatrix}.$$

If $\mathbb{N}_{\mathcal{K}}$ is a neutrosophic $\varkappa$-interior ideal in $\mathbb{N}$, then the image of $\mathbb{N}_{\mathcal{K}}$ under f is a neutrosophic $\varkappa$-interior ideal in $\mathbb{M}$.

Proof. Let $f(\mathbb{N}_{\mathcal{K}}) = \frac{\mathbb{M}}{(f(T_{\mathcal{K}}), f(I_{\mathcal{K}}), f(F_{\mathcal{K}}))}$ be the image of $\mathbb{N}_{\mathcal{K}}$ under f. Let $k_1, k_2, k_3 \in \mathbb{M}$. Then $f^{-1}(k_1) \neq \phi$, $f^{-1}(k_2) \neq \phi$ and $f^{-1}(k_3) \neq \phi$. Then there exist $s \in f^{-1}(k_1), t \in f^{-1}(k_2)$ and $u \in f^{-1}(k_3)$ such that

$$T_{\mathcal{K}}(s) = \bigwedge_{z \in f^{-1}(k_1)} T_{\mathcal{K}}(z), \quad T_{\mathcal{K}}(t) = \bigwedge_{w \in f^{-1}(k_2)} T_{\mathcal{K}}(w), \quad T_{\mathcal{K}}(u) = \bigwedge_{v \in f^{-1}(k_3)} T_{\mathcal{K}}(v),$$
$$I_{\mathcal{K}}(s) = \bigvee_{z \in f^{-1}(k_1)} I_{\mathcal{K}}(z), \quad I_{\mathcal{K}}(t) = \bigvee_{w \in f^{-1}(k_2)} I_{\mathcal{K}}(w), \quad I_{\mathcal{K}}(u) = \bigvee_{v \in f^{-1}(k_3)} I_{\mathcal{K}}(v),$$
$$F_{\mathcal{K}}(s) = \bigwedge_{z \in f^{-1}(k_1)} F_{\mathcal{K}}(z), \quad F_{\mathcal{K}}(t) = \bigwedge_{w \in f^{-1}(k_2)} F_{\mathcal{K}}(w), \quad F_{\mathcal{K}}(u) = \bigwedge_{v \in f^{-1}(k_3)} F_{\mathcal{K}}(v).$$

Now,

$$\begin{aligned}
f(T_{\mathcal{K}})(k_1k_3k_2) &= \bigwedge_{x \in f^{-1}(k_1k_3k_2)} T_{\mathcal{K}}(x) \\
&\leq T_{\mathcal{K}}(sut) \\
&\leq T_{\mathcal{K}}(u) \\
&= \bigwedge_{v \in f^{-1}(k_3)} T_{\mathcal{K}}(v) \\
&= f(T_{\mathcal{K}})(k_3), \\
f(I_{\mathcal{K}})(k_1k_3k_2) &= \bigvee_{x \in f^{-1}(k_1k_3k_2)} I_{\mathcal{K}}(x) \\
&\geq I_{\mathcal{K}}(sut) \\
&\geq I_{\mathcal{K}}(u) \\
&= \bigvee_{v \in f^{-1}(k_3)} I_{\mathcal{K}}(v) \\
&= f(I_{\mathcal{K}})(k_3), \\
f(F_{\mathcal{K}})(k_1k_3k_2) &= \bigwedge_{x \in f^{-1}(k_1k_3k_2)} F_{\mathcal{K}}(x) \\
&\leq F_{\mathcal{K}}(sut) \\
&\leq F_{\mathcal{K}}(u) \\
&= \bigwedge_{v \in f^{-1}(k_3)} F_{\mathcal{K}}(v) \\
&= f(F_{\mathcal{K}})(k_3).
\end{aligned}$$

Let $k_1, k_2 \in \mathbb{M}$ with $k_1 \le k_2$. Then

$$f(T_{\mathcal{K}})(k_1) = \bigwedge_{x \in f^{-1}(k_1)} T_{\mathcal{K}}(x)$$
$$= T_{\mathcal{K}}(s)$$
$$\le T_{\mathcal{K}}(t)$$
$$= \bigwedge_{v \in f^{-1}(k_2)} T_{\mathcal{K}}(v)$$
$$= f(T_{\mathcal{K}})(k_2),$$
$$f(I_{\mathcal{K}})(k_1) = \bigvee_{x \in f^{-1}(k_1)} I_{\mathcal{K}}(x)$$
$$= I_{\mathcal{K}}(s)$$
$$\ge I_{\mathcal{K}}(t)$$
$$= \bigvee_{v \in f^{-1}(k_2)} I_{\mathcal{K}}(v)$$
$$= f(I_{\mathcal{K}})(k_2),$$
$$f(F_{\mathcal{K}})(k_1) = \bigwedge_{x \in f^{-1}(k_1)} F_{\mathcal{K}}(x)$$
$$= F_{\mathcal{K}}(s)$$
$$\le F_{\mathcal{K}}(t)$$
$$= \bigwedge_{v \in f^{-1}(k_2)} F_{\mathcal{K}}(v)$$
$$= f(F_{\mathcal{K}})(k_2).$$

Hence, by Theorem 3.23, $f(\mathbb{N}_{\mathcal{K}})$ is a neutrosophic $\varkappa$-interior ideal in $\mathbb{M}$.

4 Conclusion

In ordered semigroups, the concepts of neutrosophic $\varkappa$-ideals, neutrosophic $\varkappa$-bi-ideals, and neutrosophic $\varkappa$-interior ideals were introduced and their properties were investigated. We defined ordered semigroups by employing various neutrosophic $\varkappa$-ideals, neutrosophic $\varkappa$-bi-ideals, neutrosophic $\varkappa$-interior ideals and so on. In our future work, we intend to define different types of notions in neutrosophic $\varkappa$-structures over-ordered semigroups, such as neutrosophic $\varkappa$-prime, neutrosophic $\varkappa$-quasi-prime, and investigate the structural properties of ordered semigroups using the concepts and results in ordered semigroups. Hopefully, our research work will continue in this direction and will create a platform for other algebraic structures.

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