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ARTICLE

Properties of Certain Subclasses of Analytic Functions Involving q-Poisson Distribution

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ABSTRACT

By using the basic (or q)-Calculus many subclasses of analytic and univalent functions have been generalized and studied from different viewpoints and perspectives. In this paper, we aim to define certain new subclasses of an analytic function. We then give necessary and sufficient conditions for each of the defined function classes. We also study necessary and sufficient conditions for a function whose coefficients are probabilities of q-Poisson distribution. To validate our results, some known consequences are also given in the form of Remarks and Corollaries.

KEYWORDS

Analytic function; q-difference operator; sufficient condition; q-Poisson distribution

1 Introduction, Basic Definitions and Motivation

A function f is said to be in the class $\mathscr{A}$ if the following normalization condition:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1$$

in the open unit disk:

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

is satisfied and having series representation:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{U}).$$

Note that the notation $\mathbb{C}$ denote the set of all complex numbers. Also, let all those functions which are univalent in $\mathscr{S}$ are placed in functions class $\mathscr{A}$.

Furthermore, let us consider a subclass $\mathscr{T}$ of the function class $\mathscr{S}$, having the following series representation:

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \qquad (z \in \mathbb{U}). \tag{1}$$

Definition 1.1. [1] A function f is said to be in the class $\mathscr{F}(\mu,\rho)$, if it satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{\mu zf'(z) + (1-\mu) f(z)}\right) \geq \rho \qquad (0 \leq \mu, \rho < 1).$$

Definition 1.2. [1] A function f is said to be in the class $\mathscr{H}(\mu,\rho)$, if it satisfy the following inequality:

$$\Re\left(\frac{f'(z) + zf''(z)}{f'(z) + \mu zf''(z)}\right) \geq \rho \qquad (0 \leq \mu, \rho < 1).$$

It should be noted that for $\mu = 0$, we have

$$\mathscr{F}(0,\rho) = \mathscr{F}(\rho) \qquad \text{and} \qquad \mathscr{H}(0,\rho) = \mathscr{H}(\rho) \qquad (0 \leq \rho < 1),$$

where $\mathscr{F}(\rho)$ and $\mathscr{H}(\rho)$ are the functions classes which were systematically studied by Silverman [2].

For a parameter ψ, if the values $0, 1, 2, 3, \cdots$ with probabilities $e^{-\psi}, \frac{\psi e^{-\psi}}{1!}, \frac{\psi^2 e^{-\psi}}{2!}, \frac{\psi^3 e^{-\psi}}{3!} \cdots$ are taken by a variable say $\mathscr{N}$, then this variable $\mathscr{N}$ is said to have Poisson distribution. Thus

$$P_i(\mathscr{N} = i) = \frac{\psi^i e^{-\psi}}{i!} \qquad (i = 0, 1, 2, 3, \cdots).$$

The Poisson distribution is a statistical distribution that calculates the probability of a certain number of events occurring in a particular time period. The Poisson distribution is commonly used to model rate of random events that occur (arrive) in some fixed time interval. The Poisson distribution models packet arrival times as an independent identically distributed process with an exponential distribution. However, it has been demonstrated in reality that packet inter-arrival durations do not follow an exponential distribution, resulting in a considerable increase in the error caused by modeling them as a Poisson distribution. User-initiated TCP (Transmission Control Protocol) session arrivals, such as remote login and file transfer, are well-modeled as Poisson processes with fixed hourly rates, but other connection arrivals deviate significantly from Poisson; modeling TELNET (Teletype Network Protocol) packet inter arrivals as exponential greatly underestimates the burstiness of TELNET traffic, according to studies.

The power series $\Upsilon(\psi, z)$ given by

$$\Upsilon(\psi, z) = z + \sum_{k=2}^{\infty} \frac{\psi^{k-1} e^{-\psi}}{(k-1)!} z^k \qquad (\forall z \in \mathbb{U} \quad \text{and} \quad \psi > 0) \tag{2}$$

with its coefficients as probabilities of Poisson distribution which was introduced by Porwal [3] (see also [4–6]). The radius of convergence of $\Upsilon(\psi, z)$ is obviously infinity which can use proved by ratio test. Porwal [3] also defined and introduced the following series:

$$\mathscr{G}(\psi, z) = 2z - \Upsilon(\psi, z) = z - \sum_{k=2}^{\infty} \frac{\psi^{k-1} e^{-\psi}}{(k-1)!} z^k \qquad (\forall z \in \mathbb{U} \quad \text{and} \quad \psi > 0). \tag{3}$$

More about special functions and related topics, we may refer to [7–11].

The Basic (or q-) series and basic (or q-) polynomials, particularly basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, are useful in a wide range of fields, including, for example, Non-Linear Electric Circuit Theory, Finite Vector Spaces, Combinatorial Analysis, Quantum Mechanics, Particle Physics, Mechanical Engineering, Lie Theory, Theory of Heat Conduction, Cosmology and Statistics (see for detail [12]).

In 1748 Euler studied a generating function for p_n, the number of partitions of a positive integer n into positive integers by considering the infinite product.

$$(q;q)_\infty^{-1} = \prod_{k=0}^{\infty} \left(1 - q^{k+1}\right)^{-1},$$

which laid foundation of the study of basic hypergeometric series (also called q-hypergeometric series or q-series). Nevertheless it took a century to get the status of independent subject after the Heine's conversion of a straightforward observation that

$$\lim_{q \to 1-} \left(\frac{1-q^\alpha}{1-q}\right) = \alpha$$

into a systematic theory of ${}_2\Phi_1$ basic hypergeometric series which is nothing but parallel to the theory of Gauss' ${}_2F_1$ hypergeometric series.

Beside from the influential study of Rogers and Thomae the subject stayed moderately comatose throughout the latter part of the nineteenth century up till Jackson embarked on a long lasting program of developing the theory of basic hypergeometric series in an organized mode (see [13] and [14]).

We next recollect certain elementary and useful concept details of the q-difference calculus, we let throughout the paper that

$$(0 < q < 1) \quad \text{and} \quad (0 \leq \mu, \rho < 1).$$

Definition 1.3. A q-generalization, q-extension, q-analogue or a q-deformation of an arbitrary number λ is given by

$$[\lambda]_q = \begin{cases} \dfrac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C} \setminus \{0\}) \\[2ex] \displaystyle\sum_{k=0}^{j-1} q^k = 1 + q + q^2 + \cdots + q^{j-1} & (\lambda = j \in \mathbb{N}) \\[2ex] 0 & (\lambda = 0). \end{cases}$$

Definition 1.4. Let the q-factorial $[n]_q!$ with $q \in (0,1)$ is define by

$$[n]_q! = \begin{cases} 1 & (n=0) \\ \prod_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 1.5. In a given subset of the set $\mathbb{C}$ of complex numbers, for function $f \in \mathscr{A}$, the q-derivative (or q-difference) operator D_q is defined by (see [13] and [14])

$$\left(D_q f\right)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases} \tag{4}$$

provided that $f'(0)$ exists.

In a given subset of $\mathbb{C}$ we can easily see from (4) that

$$\lim_{q \to 1-} \left(D_q f\right)(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z),$$

whereas f is differentiable function. Furthermore, from (1) and (4) we obtain

$$\left(D_q f\right)(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}. \tag{5}$$

In Geometric Function Theory of Complex Analysis, the role of the q-difference (or the q-derivative) operator D_q is remarkably significant. In his article published by Ismail et al. (see [15]), presented the q-deformation of the class of $\mathscr{S}^*$ of starlike functions. However, historically speaking, in 1989, in the context of Geometric Function Theory, the usage of the q-difference (or the q-derivative) operator D_q were systematically given by Srivastava [16].

More recently, Srivastava published a review article [12], in which the applications of the D_q operator in Geometric function Theory were survived. In the development of Geometric Function Theory, the works of Srivastava [16] and Ismail et al. [15] further motivate the researchers to give their finding to this field. For example, it were Wongsaijai et al. [17], who significantly studied certain subclasses of q-starlike functions. In particular, they studied the inclusion results, radius problems and certain sufficient conditions for their defined functions classes. More recently, the works of Wongsaijai et al. [17] have been generalized by Srivastava et al. (see [18,19]) in a systematic way. In fact Srivastava et al. (see [18,19]) make use of the q-Calculus and certain Janwoski functions in order to devolved their results, which essentially are the generalizations of [17]. Some more recent works related to q-calculus can be found in [20–28].

Motivated by the above-mentioned works of Srivastava [12,16] and Ismail et al. [15], in this article we shall mainly generalize the works presented in [1] and [3]. We shall study a number of sufficient conditions here. A necessary and sufficient condition shall be encounter for a certain function $\mathscr{G}_q(\psi, z) \in \mathscr{F}_q(\mu, \rho)$ whose coefficients are nothing but merely the probabilities of q-Poisson distribution. In order to validate our results many new and known consequences are also included.

2 The Subclasses $\mathscr{F}_q(\mu,\rho)$ and $\mathscr{H}_q(\mu,\rho)$ of Analytic Functions

In this section, making use of the concept of q-calculus and the aforementioned works, we first define certain new subclasses $\mathscr{F}_q(\mu,\rho)$ and $\mathscr{H}_q(\mu,\rho)$ of analytic functions which involve the q-calculus. We then give certain sufficient conditions and closure results for these defined function classes. We have also highlighted some known and new consequences of our main results. The class $\mathscr{F}_q(\mu,\rho)$ can be define as:

Definition 2.1. A function f with series representation (1) is to be placed in the class $\mathscr{F}_q(\mu,\rho)$ if

$$\Re\left(\frac{z\left(D_q f\right)(z)}{\mu z\left(D_q f\right)(z)+(1-\mu) f(z)}\right)\geq \rho.$$

Remark 1. First of all, it is easy to see that, if we put $\mu=0$, we have

$$\mathscr{F}_q(0,\rho)=\mathscr{F}_q(\rho)=\mathscr{T}\mathscr{S}_q^*(\rho),$$

where $\mathscr{T}\mathscr{S}_q^*(\rho)$ is the function class, introduced and studied in [17]. Secondly, we have

$$\lim_{q\to 1-}\mathscr{F}_q(\mu,\rho)=\mathscr{F}(\mu,\rho),$$

where $\mathscr{F}(\mu,\rho)$ is defined and studied in [1]. Thirdly, we see that

$$\lim_{q\to 1-}\mathscr{F}_q(0,\rho)=\mathscr{F}(\rho)=\mathscr{T}\mathscr{S}^*(\rho),$$

where $\mathscr{T}\mathscr{S}^*(\rho)$ function class is defined and studied by Silverman [2].

Definition 2.2. A function f with series representation (1) is to be placed in the class $\mathscr{H}_q(\mu,\rho)$ if

$$\Re\left(\frac{\left(D_q f\right)(z)+z\left(D_q\left(D_q f\right)\right)(z)}{\left(D_q f\right)(z)+\mu z\left(D_q\left(D_q f\right)\right) f(z)}\right)\geq \rho.$$

2.1 Sufficient Conditions for the Class $\mathscr{F}_q(\mu,\rho)$ and $\mathscr{H}_q(\mu,\rho)$

Theorem 2.1. A function f of the form (1) is in the class $\mathscr{F}_q(\mu,\rho)$ if and only if

$$\sum_{k=2}^{\infty}\left([k]_q-\mu\rho[k]_q-\rho+\mu\rho\right)|a_k|\leq(1-\rho). \tag{6}$$

For the following function $f_1(z)$ the result is best possible

$$f_1(z)=z-\frac{1-\rho}{[k]_q-\mu\rho[k]_q-\rho+\mu\rho}z^k \qquad (k\geq 2).$$

Proof. If we assume that $f\in\mathscr{F}_q(\mu,\rho)$, then we have

$$\Re\left(\frac{z\left(D_q f\right)(z)}{\mu z\left(D_q f\right)(z)+(1-\mu) f(z)}\right)$$

$$= \Re\left(\frac{z - \sum\limits_{k=2} [k]_q |a_k| z^k}{z - \sum\limits_{k=2} \left(\mu [k]_q + (1-\mu)\right) |a_k| z^k}\right)$$

$$\geq \rho.$$

If the complex number z lie on the real axis side, then

$$\frac{z\left(D_q f\right)(z)}{\mu z\left(D_q f\right)(z) + (1-\mu) f(z)}$$

is real. And so in limiting case if we take $z \to 1-$, along the real line, then, we have

$$\frac{1 - \sum\limits_{k=2} [k]_q |a_k|}{1 - \sum\limits_{k=2} \left(\mu [k]_q + (1-\mu)\right) |a_k|} \geq \rho.$$

We see that the last inequality is equivalent to (6).

Conversely, suppose that (6) hold true. Then adding

$$-(1-\rho) \sum_{k=2} \left(\mu [k]_q + (1-\mu)\right) |a_k|$$

to both sides of (6), we get

$$(1-\mu) \sum_{k=2} \left([k]_q - 1\right) |a_k| \leq (1-\rho) \left(1 - \sum_{k=2} \left(\mu [k]_q + (1-\mu)\right) |a_k|\right). \tag{7}$$

On the other hand, we see that

$$\left|\frac{z\left(D_q f\right)(z)}{\mu z\left(D_q f\right)(z) + (1-\mu) f(z)} - 1\right|$$

$$= \left|\frac{(1-\mu) \sum\limits_{k=2} \left([k]_q - 1\right) |a_k| z^{k-1}}{1 - \sum\limits_{k=2} \left(\mu [k]_q + (1-\mu)\right) |a_k| z^{k-1}}\right|$$

$$\leq \frac{(1-\mu) \sum\limits_{k=2} \left([k]_q - 1\right) |a_k|}{1 - \sum\limits_{k=2} \left(\mu [k]_q + (1-\mu)\right) |a_k|}. \tag{8}$$

It is now easy to see from (7) that the last expression in (8) is bounded above by $(1-\rho)$. Which evidently showing that $f \in \mathscr{F}_q(\mu, \rho)$.

Corollary 1. If $f \in \mathscr{F}_q(\mu, \rho)$ then, we have

$$|a_k| \leq \frac{1-\rho}{[k]_q - \mu\rho [k]_q - \rho + \mu\rho} \qquad (k \geq 2).$$

Remark 2. First of all, we see that if we put $\mu = 0$, in Theorem 2.1, we get the known result due to Wongsaijai et al. (see [17]). Secondly if we put $\mu = 0$ and let $q \to 1-$, Theorem 2.1, we have the sufficient condition for the class $\mathscr{S}^*(\rho)$, which was studied by Silverman [2]. Thirdly, if we let $q \to 1-$, Theorem 2.1 coincide with a known result given in [1].

Theorem 2.2. A function f of the form (1) is in the class $\mathscr{H}_q(\mu,\rho)$ if and only if

$$\sum_{k=2}^{\infty} [k]_q \left([k]_q - \mu\rho\,[k]_q - \rho + \mu\rho\right) |a_k| \leq (1-\rho)\,. \tag{9}$$

The equality holds for the function $f_1(z)$ which is given by

$$f_1(z) = z - \frac{1-\rho}{[k]_q \left([k]_q - \mu\rho\,[k]_q - \rho + \mu\rho\right)} z^k \qquad (k \geq 2)\,.$$

Proof. This theorem can be proved by using the arguments similar which we used in Theorem 2.1.

2.2 *Closure Theorems for the Class* $\mathscr{F}_q(\mu,\rho)$

We define the function $f_\varkappa(z) \quad (\varkappa = 1,2,3,\ldots,l)$ as

$$f_\varkappa(z) = z - \sum_{n=2}^{\infty} \left|a_{n,\varkappa}\right| z^n \qquad (z \in \mathbb{U})\,. \tag{10}$$

Furthermore, we now give certain closure results for the function involved in (10).

Theorem 2.3 Let the functions $f_\varkappa(z) \quad (\varkappa = 1,2,3,\ldots,l)$ defined by (10) be in the class $f \in \mathscr{F}_q(\mu,\rho)$. Then the function $\mathscr{T} \in \mathscr{F}_q(\mu,\rho)$, where

$$\mathscr{T}(z) = \sum_{\varkappa=1}^{l} \Gamma_\varkappa f_\varkappa(z) \qquad \left(\Gamma_\varkappa \geq 0 \qquad \sum_{\varkappa=1}^{l} \Gamma_\varkappa = 1\right).$$

Proof. From (10), we have

$$\mathscr{T}(z) = z - \sum_{k=2}^{\infty} \left|\sum_{\varkappa=1}^{l} \Gamma_\varkappa a_{k,\varkappa}\right| z^k.$$

Now making use of Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \left([k]_q - \mu\rho\,[k]_q - \rho + \mu\rho\right) \left|\sum_{\varkappa=1}^{l} \Gamma_\varkappa a_{k,\varkappa}\right|$$

$$= \sum_{\varkappa=1}^{l} \Gamma_\varkappa \left(\sum_{k=2}^{\infty} \left([k]_q - \mu\rho\,[k]_q - \rho + \mu\rho\right) \left|a_{k,\varkappa}\right|\right)$$

$$\leq \sum_{\varkappa=1}^{l} \Gamma_{\varkappa} (1-\rho) = (1-\rho) \qquad \left(\sum_{\varkappa=1}^{l} \Gamma_{\varkappa} = 1 \right).$$

Now by Theorem 2.1, the proof of Theorem 2.3 is completed.

Theorem 2.4. The class $\mathscr{F}_q(\mu,\rho)$ defined in Definition 6 is closed under the convex combination.

Proof. To prove our result, we let that the given functions $f_{\varkappa}(z)$ $(\varkappa = 1,2)$ which are defined by (10) be in the class $\mathscr{F}_q(\mu,\rho)$. It is enough to show that

$$g(z) = \rho f_1(z) + (1-\rho) f_2(z) \qquad (0 \leq \rho \leq 1)$$

is in the class $\mathscr{F}_q(\mu,\rho)$. Since

$$g(z) = z - \sum_{k=2}^{\infty} \left(\rho a_{k,1} + (1-\rho) \left| a_{k,2} \right| \right) z^k \qquad (0 \leq \rho \leq 1).$$

By Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \left([k]_q - \mu\rho [k]_q - \rho + \mu\rho \right) \left| \rho \left(a_{k,1} \right) + (1-\rho) \left(a_{k,2} \right) \right|$$

$$\leq \sum_{k=2}^{\infty} \left([k]_q - \mu\rho [k]_q - \rho + \mu\rho \right) \left| \rho a_{k,1} \right|$$

$$+ \sum_{k=2}^{\infty} \left([k]_q - \mu\rho [k]_q - \rho + \mu\rho \right) \left| (1-\rho) a_{k,2} \right|$$

$$\leq \rho (1-\rho) + (1-\rho)(1-\rho) = (1-\rho).$$

which evidently complete the proof of Theorem 2.4.

3 The Classes $\mathscr{F}_q(\mu,\rho)$ and $\mathscr{H}_q(\mu,\rho)$ Involving q-Poisson Distribution

The Euler distribution or Heine distribution, as shown in [29] is a q-deformed Poisson distribution that is known in the literature. All of these, of course, are natural deformations in some sense, but they are all discrete. Kemp [30] demonstrated that the Euler and Heine distributions are, respectively, the limiting forms of a q-analogue of the negative binomial distribution and one of the binomial distribution. Jing [29] proposed a new type of q-analogue of the binomial distribution that uses the Euler distribution as its limiting form.

Furthermore, the theory of q-discrete distributions is quite significant because of theoretical probabilistic and statistical interest. In quantum probability it appears at Brownian motion, further from fascinating Markov chains with discrete finite or infinite state spaces it arises as a steady state distributions (see for example [29–31]). Our aim in this section is to consider a necessary and sufficient condition for a function $\mathscr{G}_q(\psi,z) \in \mathscr{F}_q(\mu,\rho)$ such that its coefficients are probabilities of

q-Poisson distribution. Some known consequences are also highlighted. The q-Poisson distribution $\mathcal{N}_q$ can be define as follows (see [32]):

$$P_i\left(\mathcal{N}_q = [i]_q\right) = \frac{\psi^i}{[i]_q!}\left(1-\psi\left(1-q\right)\right)^{\infty} = \frac{\psi^i}{[i]_q!}e_q^{-\psi},$$

where

$$e_q^{\psi} = \sum_{k=0}^{\infty}\frac{\psi^k}{[k]_q!}.$$

The graph of e_q^{ψ} is given in Fig. 1.

In other words, if $\mathcal{N}_q$ takes the values $0, 1, 2, 3, \cdots$ with probabilities $e_q^{-\psi}, \frac{\psi e_q^{-\psi}}{[1]!}, \frac{\psi^2 e_q^{-\psi}}{[2]!}, \frac{\psi^3 e_q^{-\psi}}{[3]!}\cdots$ for the parameter ψ respectively, then $\mathcal{N}_q$ is called q-Poisson distributed.

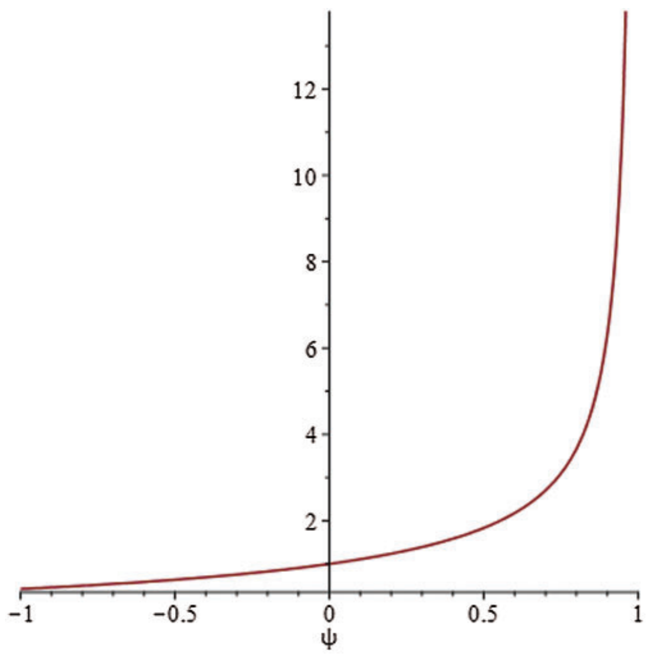

Figure 1: Graph of e_q^{ψ} with $q = 0.5$ and $-1 < \psi < 1$

In the following equations we settled a power series so that its coefficients are probabilities the of q-Poisson distribution as

$$\begin{aligned}\Upsilon_q\left(\psi, z\right) &= z + \sum_{k=2}^{\infty}\frac{\psi^{k-1}}{[k-1]_q!}\left(1-\psi\left(1-q\right)\right)^{\infty} z^k \\ &= z + \sum_{k=2}^{\infty}\frac{\psi^{k-1}}{[k-1]_q!}e_q^{-\psi} z^k \qquad \left(\forall z \in \mathbb{U} \quad \text{and} \quad \psi > 0\right).\end{aligned}$$

It could be seen that by ratio test the radius of convergence of $\Upsilon_q(\psi, z)$ is infinity. Therefore we define the following series:

$$\mathcal{G}_q(\psi, z) = 2z - \Upsilon_q(\psi, z) = z - \sum_{k=2}^{\infty} \frac{\psi^{k-1}}{[k-1]_q!} e_q^{-\psi} z^k \qquad (\forall z \in \mathbb{U} \quad \text{and} \quad \psi > 0).$$

Theorem 3.1. A function $\mathcal{G}_q(\psi, z)$ is to be placed in the functions class $\mathcal{F}_q(\mu, \rho)$ if and only if

$$q(1 - \rho\mu)\psi e_q^{\psi} < (1 - \rho) \qquad (\psi > 0). \tag{11}$$

Proof. Since

$$\mathcal{G}_q(\psi, z) = z - \sum_{k=2}^{\infty} \frac{\psi^{k-1}}{[k-1]_q!} e_q^{-\psi} z^k \qquad (\forall z \in \mathbb{U} \quad \text{and} \quad \psi > 0).$$

According to Theorem 2.1, we must show that

$$\sum_{k=2}^{\infty} \left([k]_q - \mu\rho [k]_q - \rho + \mu\rho\right) \left(\frac{\psi^{k-1}}{[k-1]_q!} e_q^{-\psi}\right) \leq (1 - \rho).$$

Therefore, we now consider

$$\begin{aligned}
&\sum_{k=2}^{\infty} \left([k]_q - \mu\rho [k]_q - \rho + \mu\rho\right) \left(\frac{\psi^{k-1}}{[k-1]_q!} e_q^{-\psi}\right) \\
&\quad = e_q^{-\psi} \sum_{k=2}^{\infty} \left(\left([k]_q - 1\right)(1 - \mu\rho) + (1 - \rho)\right) \left(\frac{\psi^{k-1}}{[k-1]_q!}\right) \\
&\quad = e_q^{-\psi} \left(q(1 - \mu\rho) \sum_{k=2}^{\infty} \frac{[k-1]_q \psi^{k-1}}{[k-1]_q!} + (1 - \rho) \sum_{k=2}^{\infty} \left(\frac{\psi^{k-1}}{[k-1]_q!}\right)\right) \\
&\quad = e_q^{-\psi} \left(q(1 - \mu\rho) \sum_{k=2}^{\infty} \frac{\psi^{k-1}}{[k-2]_q!}\right) + e_q^{-\psi}(1 - \rho) \sum_{k=1}^{\infty} \left(\frac{\psi^k}{[k]_q!}\right) \\
&\quad = e_q^{-\psi} \left(q(1 - \mu\rho)\psi \sum_{k=0}^{\infty} \frac{\psi^k}{[k]_q!}\right) + e_q^{-\psi}(1 - \rho) \sum_{k=1}^{\infty} \left(\frac{\psi^k}{[k]_q!}\right) \\
&\quad = e_q^{-\psi} \left(q(1 - \mu\rho)\psi e_q^{\psi} + (1 - \rho)\left(e_q^{\psi} - 1\right)\right).
\end{aligned} \tag{12}$$

The last expression in (12) is evidently bounded by functional $(1 - \rho)$ in the case if and only if relation given in (11) holds true.

Thus we complete the required proof of our Theorem.

Corollary 2. A function $\mathscr{G}_q(\psi, z)$ is to be placed in the functions class $\mathscr{F}_q(0,\rho) = \mathscr{T}\mathscr{S}_q^*(\rho)$ if and only if

$$q\psi e_q^{\psi} < (1-\rho) \qquad (\psi > 0). \tag{13}$$

Remark 3. If we set $q \to 1-$ in Theorem 3.1, it will lead us to the result given by Porwal [3].

Theorem 3.2. A function $\mathscr{G}_q(\psi, z)$ is to be placed in the class $\mathscr{F}_q(\mu,\rho)$ if and only if

$$q(1-\rho\mu)\psi e_q^{\psi} < (1-\rho) \qquad (\psi > 0). \tag{14}$$

Proof. Since

$$\mathscr{G}_q(\psi, z) = z - \sum_{k=2}^{\infty} \frac{\psi^{k-1}}{[k-1]_q!} e_q^{-\psi} z^k \qquad (\forall z \in \mathbb{U} \quad \text{and} \quad \psi > 0).$$

According to Theorem 2.2, we must show that

$$\sum_{k=2}^{\infty} [k]_q \left([k]_q - \mu\rho [k]_q - \rho + \mu\rho\right) \left(\frac{\psi^{k-1}}{[k-1]_q!} e_q^{-\psi}\right) \leq (1-\rho).$$

Therefore, we now consider

$$\begin{aligned}
&\sum_{k=2}^{\infty} [k]_q \left([k]_q - \mu\rho [k]_q - \rho + \mu\rho\right) \left(\frac{\psi^{k-1}}{[k-1]_q!} e_q^{-\psi}\right) \\
&= \sum_{k=2}^{\infty} \left\{\left([k]_q - 1\right)\left([k]_q - 2\right)(1-\mu\rho) \right. \\
&\qquad \left. + (3-\mu\rho-\rho)\left([k]_q - 1\right) + (1-\rho)\right\} \left(\frac{\psi^{k-1}}{[k-1]_q!} e_q^{-\psi}\right) \\
&= e_q^{-\psi} \left(q^3(1-\mu\rho) \sum_{k=3}^{\infty} \frac{\psi^{k-1}}{[k-3]_q!} + (2 - q\mu\rho - \rho + q) q \right. \\
&\qquad \left. \cdot \sum_{k=2}^{\infty} \frac{\psi^{k-1}}{[k-2]_q!} + (1-\rho) \sum_{k=2}^{\infty} \frac{\psi^{k-1}}{[k-1]_q!}\right) \\
&= e_q^{-\psi} \left(q^3\psi^2(1-\mu\rho) e_q^{\psi} + (2 - q\mu\rho - \rho + q) q\psi e_q^{\psi} + (1-\rho)\left(e_q^{\psi} - 1\right)\right) \\
&= q^3\psi^2(1-\mu\rho) + (2 - q\mu\rho - \rho + q) q\psi + (1-\rho)\left(1 - e_q^{-\psi}\right).
\end{aligned} \tag{15}$$

We in last expression see that the $(1-\rho)$ is bound of (15) if and only if (11) holds true. Thus we have completed the proof of our Theorem.

Remark 4. If in Theorem 3.2, we let $q \to 1-$, we will arrive at the result given by Porwal [3].

4 Concluding Remarks and Observations

In our present work, by using the q-calculus, we have first defined certain new subclasses of an analytic function. We then gave necessary and sufficient conditions for each of the defined function classes. We have also studied necessary and sufficient conditions for a function whose coefficients are probabilities of q-Poisson distribution. We have also given particular known and new consequences in the form of Remarks and Corollaries.

The usage of basic (or q-) series in many diverse areas of Mathematics and Physics makes it very important. By using the basic (or q-) series, some wonderful works have been done. As we described in Section 1, the Srivastava's observation [12] about the so-called (p, q)-calculus, we arrived at the point that indeed the result presented in this paper can be produced for the rather straightforward (p, q)-variations.

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