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ARTICLE

Aggregation Operators for Interval-Valued Pythagorean Fuzzy Soft Set with Their Application to Solve Multi-Attribute Group Decision Making Problem

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ABSTRACT

Interval-valued Pythagorean fuzzy soft set (IVPFSS) is a generalization of the interval-valued intuitionistic fuzzy soft set (IVIFSS) and interval-valued Pythagorean fuzzy set (IVPFS). The IVPFSS handled more uncertainty comparative to IVIFSS; it is the most significant technique for explaining fuzzy information in the decision-making process. In this work, some novel operational laws for IVPFSS have been proposed. Based on presented operational laws, two innovative aggregation operators (AOs) have been developed such as interval-valued Pythagorean fuzzy soft weighted average (IVPFSWA) and interval-valued Pythagorean fuzzy soft weighted geometric (IVPFSWG) operators with their fundamental properties. A multi-attribute group decision-making (MAGDM) approach has been established utilizing our developed operators. A numerical example has been presented to ensure the validity of the proposed MAGDM technique. Finally, comparative studies have been given between the proposed approach and some existing studies. The obtained results through comparative studies show that the proposed technique is more credible and reliable than existing approaches.

KEYWORDS

Interval-valued Pythagorean fuzzy soft set; IVPFSWA operator; IVPFSWG operator; MAGDM

1 Introduction

MAGDM is considered as the most appropriate technique to find the finest alternative from all possible alternatives, following criteria or attributes. Conventionally, it is supposed that all information that accesses the alternative in terms of attributes and their corresponding weights are articulated in the form of crisp numbers. On the other hand, in real-life circumstances, most of the decisions are taken in situations where the objectives and limitations are usually indefinite or ambiguous. To overcome such ambiguities and anxieties, Zadeh offered the notion of the fuzzy set (FS) [1], a prevailing tool to handle the obscurities and uncertainties in decision making (DM). Such a set allocates to all objects a membership value ranging from 0 to 1. Mostly, experts consider membership and a non-membership value in the DM process which cannot be handled by FS. Atanassov [2] introduced the idea of the intuitionistic fuzzy set (IFS) to overcome the aforementioned limitation. In 2011, Wang et al. [3] presented numerous operations on IFS, such as Einstein product, Einstein sum, etc., and constructed some novel AOs. They also discussed some important properties of these operators and utilized their proposed operators to resolve multi-attribute decision making (MADM). Atanassov [4] presented a generalized form of IFS in the light of ordinary interval values, called interval-valued intuitionistic fuzzy set (IVIFS). Garg et al. [5] extended the concept of IFS and presented a novel concept of the cubic intuitionistic fuzzy set (CIFS) which is a successful tool to represent vague data by embedding both IFS and IVIFS directly. They also discussed several desirable properties of CIFS.

The above-mentioned models have been well-recognized by the specialists but the existing IFS is unable to handle the inappropriate and vague data because it is considered to envision the linear inequality between the membership and non-membership grades. For example, if decision-makers choose membership and non-membership values 0.9 and 0.6 respectively, then the above-mentioned IFS theory is unable to deal with it because $0.9 + 0.6 \geq 1$. To resolve the aforesaid limitation, Yager [6] presented the idea of the Pythagorean fuzzy set (PFS) by amending the basic condition $\kappa + \delta \leq 1$ to $\kappa^2 + \delta^2 \leq 1$ and developed some results associated with score function and accuracy function. Rahman et al. [7] developed the Pythagorean fuzzy Einstein weighted geometric operator and presented a MAGDM methodology utilizing their proposed operator. Zang et al. [8] developed some basic operational laws and prolonged the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) method to resolve multi-criteria decision-making (MCDM) complications for PFS information. Wei et al. [9] offered the Pythagorean fuzzy power AOs along with basic characteristics, they also established a DM technique to resolve MADM difficulties based on presented operators. Wang et al. [10] offered the interaction operational laws for Pythagorean fuzzy numbers (PFNs) and developed power Bonferroni mean operators. To assess the professional health risk, IIbahar et al. [11] offered the Pythagorean fuzzy proportional risk assessment technique. Zhang [12] proposed a novel DM approach based on similarity measures to resolve multi-criteria group decision making (MCGDM) difficulties for the PFS.

All of the aforementioned techniques have a wide range of applications, but owing to their ineffectiveness, they have several restrictions with the parameterization tool. Presenting a solution to this type of uncertainty and obfuscation Molodtsov [13] established the idea of soft sets (SS) and described some basic operations with their characteristics to handle the above-mentioned confusion and ambiguity. Maji et al. [14] expanded the concept of SS and developed many basic and binary operations for it. Maji et al. [15] developed the fuzzy soft set with some desirable properties by merging two existing notions FS and SS. Maji et al. [16] developed the notion of the intuitionistic fuzzy soft set (IFSS) and some fundamental operations with their necessary properties. Garg et al. [5] presented the cubic IFSS and established some AOs for cubic IFSS. They also planned a DM technique based on their developed operators. Zulqarnain et al. [17] planned the TOPSIS method based on the correlation coefficient for interval-valued IFSS to solve MADM problems. Jiang et al. [18] introduced the notion of the interval-valued intuitionistic fuzzy soft set (IVIFSS) and discussed some of their basic properties. Narayanamoorthy et al. [19]

proposed the score function for a normal wiggly hesitant fuzzy set and utilized it to expose the deepest ideas hidden in the thought-level of the decision-makers. Narayanamoorthy et al. [20] introduced the hesitant fuzzy subjective and objective weight integrated method to find weights under hesitant fuzzy information. They also presented a novel ranking methodology for hesitant fuzzy sets. Ramya et al. [21] developed the interval-valued hesitant Pythagorean fuzzy set under the normal wiggly mathematical methodology and used it to solve the MCDM problem. Peng et al. [22] merged two well-known theories PFS and SS and offered the concept of Pythagorean fuzzy soft set (PFSS). Zulqarnain et al. [23] developed the AOs for PFSS with their application for the green supplier chain management. Zulqarnain et al. [24] introduced an advanced form of AOs considering the interaction and construct a DM approach based on their developed interactive AOs. Smarandache [25] prolonged the idea of SS to hypersoft sets (HSS) by substituting the single-parameter function f with a multi-parameter (sub-attribute) function. He privileges that HSS proficiently contracts with inexact data comparative to SS.

MAGDM is a very effective and well-known tool to examine fuzzy data more effectively. Therefore, it is obvious from the published literature that the interval-valued structures are more general and increase more consideration in decision-making difficulties. The choice of vehicle is a key part of real-life and will advise on complications of MAGDM. Lack of thinking about the ambiguity of alternative associations will be the core motivation for some MAGDM concerns about the undesirable consequences. By using a wealth of existing content, it contains previous criticisms and suppressed sensitivities. Many logical and scientific tools/procedures are offended in the literature for choosing the most suitable vehicle. As far as we know, there is currently no work on the AOs of IVPFSS. Therefore, this article proposes some operational laws for interval-valued Pythagorean fuzzy soft numbers (IVPFSN). The presented IVPFSN is well worth observing the inaccurate information that occurs in the complications of daily life. Therefore, the main purpose of this work is to propose new IVPFSWA and IVPFSWG operators based on the established operational laws. An algorithm based on the proposed operators to solve the decision-making problem is proposed. To prove the effectiveness of the proposed decision-making method, we use a numerical example to illustrate it. The main benefit of the proposed operator is that the proposed operator can reduce to IVIFSS and IVFSS operators under some specific conditions of unconfidence. The organization of this paper is given as follows: Section 2 of this paper consists of some basic concepts which help us to develop the structure of the following research. In Section 3, some novel operational laws for IVPFSN have been proposed. Also, in the same section, IVPFSWA and IVPFSWG operators have been introduced based on our developed operators with their basic properties. In Section 4, a MAGDM approach has been constructed based on the proposed AOs. To ensure the practicality of the developed approach a numerical example has been presented for the selection of the best vehicle in Section 5.

2 Preliminaries

This section consists of some basic definitions which will provide a structure to form the following work.

Definition 2.1 [1]

Let U be a collection of objects then a fuzzy set (FS) A over U is defined as

$$A = \{(t, \kappa(t)) \mid t \in U\}$$

where, $\kappa_A(t): X \to [0, 1]$ is a membership grade function.

Definition 2.2 [26]

Let U be a collection of objects then an interval-valued fuzzy set (IVFS) A over U is defined as

$$A = \left\{ \left(t, \left[\kappa^{l}(t), \kappa^{u}(t) \right] \right) \middle| t \in U \right\}$$

where, $\kappa^{l}(t), \kappa^{u}(t) \in [0, 1]$ and represents the lower and upper bounds of the membership value.

Definition 2.3 [4]

Let U be a collection of objects then an interval-valued intuitionistic fuzzy set (IVIFS) A over U is defined as

$$A = \left\{ \left(t, \left(\left[\kappa_A^{l}(t), \kappa_A^{u}(t) \right], \left[\delta_A^{l}(t), \delta_A^{u}(t) \right] \right) \right) | t \in U \right\}$$

where, $\left[\kappa_A^{l}(t), \kappa_A^{u}(t) \right]$ and $\left[\delta_A^{l}(x), \delta_A^{u}(x) \right]$ are intervals for membership and non-membership grades, respectively, whereas $\left[\kappa_A^{l}(t), \kappa_A^{u}(t) \right] and \left[\delta_A^{l}(t), \delta_A^{u}(t) \right] \subseteq [0, 1]$, $0 \le \kappa_A^{l}(t), \kappa_A^{u}(t), \delta_A^{l}(t), \delta_A^{u}(t) \le 1$, and $0 \le \kappa_A^{u}(x) + \delta_A^{u}(x) \le 1$.

Definition 2.4 [27]

Let U be a collection of objects then an interval-valued Pythagorean fuzzy set (IVPFS) A over U is defined as

$$A = \left\{ \left(x, \left(\left[\kappa_A^{l}(t), \kappa_A^{u}(t) \right], \left[\delta_A^{l}(t), \delta_A^{u}(t) \right] \right) \right) | t \in U \right\}$$

where, $\left[\kappa_A^{l}(t), \kappa_A^{u}(t) \right]$ and $\left[\delta_A^{l}(t), \delta_A^{u}(t) \right]$ represents the intervals for membership and non-membership grades, respectively. Furthermore, $0 \le \left(\kappa_A^{u}(t) \right)^2 + \left(\delta_A^{u}(t) \right)^2 \le 1$ and $\left[\kappa_A^{l}(t), \kappa_A^{u}(t) \right] \subseteq [0, 1]$ and $\left[\delta_A^{l}(t), \delta_A^{u}(t) \right] \subseteq [0, 1]$.

Definition 2.5 [13]

Let U be a universal set and $\mathbb{N} = \{t_1, t_2, t_3, \ldots, t_m\}$ be set of attributes then a pair $(F, \mathbb{N})$ is called a soft set (SS) over U where $F \colon \mathbb{N} \to K^U$ is a mapping and K^U is known as a collection of all subsets of universal set U.

Definition 2.6 [19]

Let U be a universal set and $\mathbb{N}$ be a set of attributes then a pair $(\Omega, \mathbb{N})$ is called an interval-valued intuitionistic fuzzy soft set (IVIFSS) over U. Where $\Omega \colon \mathbb{N} \to IK^U$ is a mapping and IK^U is known as a collection of all interval-valued intuitionistic fuzzy subsets of universal set U and $A \subset \mathbb{N}$.

$$(\Omega, A) = \left\{ t, \left(\left[\kappa_A^{l}(t), \kappa_A^{u}(t) \right], \left[\delta_A^{l}(t), \delta_A^{u}(t) \right] \right) | t \in A \right\}$$

where, $\left[\kappa_A^{l}(t), \kappa_A^{u}(t) \right] and \left[\delta_A^{l}(t), \delta_A^{u}(t) \right]$ are intervals for membership grade and non-membership functions respectively with $0 \le \kappa_A^{u}(t) + \delta_A^{u}(t) \le 1$.

Definition 2.7

Let U be a universal set and $\mathbb{N}$ be set of attributes then a pair $((\Omega, \mathbb{N})$ is called an interval-valued Pythagorean fuzzy soft set (IVPFSS) over U where $\Omega\colon \mathbb{N} \to \wp K^U$ is a mapping and $\wp K^U$ is known as the collection of all interval-valued Pythagorean fuzzy subsets of universal set U.

$$(\Omega, A) = \left\{t, \left(\left[\kappa_A^l(t), \kappa_A^u(t)\right], \left[\delta_A^l(t), \delta_A^u(t)\right]\right) | t \in A\right\}$$

where, $\left[\kappa_A^l(t), \kappa_A^u(t)\right], \left[\delta_A^l(t), \delta_A^u(t)\right]$ are intervals for membership grade and non-membership grade, respectively with $0 \le \left(\kappa_A^u(t)\right)^2 + \left(\delta_A^u(t)\right)^2 \le 1$ and $A \subset \mathbb{N}$.

Definition 2.8

Let $\mathcal{M}_e = \left(\left[\kappa^l, \kappa^u\right], \left[\delta^l, \delta^u\right]\right)$ be an interval-valued Pythagorean fuzzy soft number (IVPFSN), then the score function is defined as follows:

$$S(\mathcal{M}_e) = \frac{\left(\kappa^l\right)^2 + (\kappa^u)^2 - \left(\delta^l\right)^2 - (\delta^u)^2}{2}$$

Definition 2.9

Let $\mathcal{M}_e = \left(\left[\kappa^l, \kappa^u\right], \left[\delta^l, \delta^u\right]\right)$ be an IVPFSN, then accuracy function is defined as follows:

$$S(\mathcal{M}_e) = \frac{\left(\kappa^l\right)^2 + (\kappa^u)^2 + \left(\delta^l\right)^2 + (\delta^u)^2}{2}$$

3 Aggregation Operators for Interval Valued Pythagorean Fuzzy Soft Sets

In this section, we are going to define operational laws under IVPFSNs. Based on these operational laws, we shall also present interval-valued Pythagorean fuzzy soft weighted average (IVPFSWA) and interval-valued Pythagorean fuzzy soft geometric (IVPFSWG) operators.

3.1 Operational Laws for Interval Valued Pythagorean Fuzzy Soft Numbers

Let $\mathcal{M}_e = \left(\left[\kappa^l, \kappa^u\right], \left[\delta^l, \delta^u\right]\right)$, $\mathcal{M}_{e_{11}} = \left(\left[\kappa_{11}^l, \kappa_{11}^u\right], \left[\delta_{11}^l, \delta_{11}^u\right]\right)$, and

$\mathcal{M}_{e_{12}} = \left(\left[\kappa_{12}^l, \kappa_{12}^u\right], \left[\delta_{12}^l, \delta_{12}^u\right]\right)$ be three interval-valued Pythagorean fuzzy soft numbers and β be a positive real number, and by algebraic norms, we have

(1) $\mathcal{M}_{e_{11}} \oplus \mathcal{M}_{e_{12}} = \left(\left[\sqrt{\kappa_{11}^{l2} + \kappa_{12}^{l2} - \kappa_{11}^{l2}\kappa_{12}^{l2}}, \sqrt{\kappa_{11}^{u2} + \kappa_{12}^{u2} - \kappa_{11}^{u2}\kappa_{12}^{u2}}\right], \left[\delta_{11}^l\delta_{12}^l, \delta_{11}^u\delta_{12}^u\right]\right)$

(2) $\mathcal{M}_{e_{11}} \otimes \mathcal{M}_{e_{12}} = \left(\left[\kappa_{11}^l\kappa_{12}^l, \kappa_{11}^u\kappa_{12}^u\right], \left[\sqrt{\delta_{11}^{l2} + \delta_{12}^{l2} - \delta_{11}^{l2}\delta_{12}^{l2}}, \sqrt{\delta_{11}^{u2} + \delta_{12}^{u2} - \delta_{11}^{u2}\delta_{11}^{u2}}\right]\right)$

(3) $\beta\mathcal{M}_e = \left(\left[\sqrt{1 - \left(1 - \kappa^{l2}\right)^{\beta}}, \sqrt{1 - \left(1 - \kappa^{u2}\right)^{\beta}}\right], \left[\delta^{l\beta}, \delta^{u\beta}\right]\right)$

$$= \left(\sqrt{1 - \left(1 - \left[\kappa^l, \kappa^u\right]^2\right)^{\beta}}, \left[\delta^{l\beta}, \delta^{u\beta}\right]\right)$$

$$(4)\ \mathcal{M}_e{}^{\beta} = \left(\left[\kappa^{l\beta}, \kappa^{u\beta}\right], \left[\sqrt{1-\left(1-\delta^{l2}\right)^{\beta}}, \sqrt{1-\left(1-\delta^{u2}\right)^{\beta}}\right]\right)$$
$$= \left(\left[\kappa^{l\beta}, \kappa^{u\beta}\right], \sqrt{1-\left(1-\left[\delta^{l}, \delta^{u}\right]^{2}\right)^{\beta}}\right)$$

3.2 Interval Valued Pythagorean Fuzzy Soft Weighted Average Operator

Let $\mathcal{M}_{e_{ij}} = \left(\left[\kappa^l_{ij}, \kappa^u_{ij}\right], \left[\delta^l_{ij}, \delta^u_{ij}\right]\right)$ be a collection of interval-valued Pythagorean fuzzy soft numbers (IVPFSNs), and ω_i and ν_j are the weight vector for experts and parameters, respectively, with given conditions $\omega_i > 0, \sum_{i=1}^{n} \omega_i = 1; \nu_j > 0, \sum_{j=1}^{m} \nu_j = 1$. Then, the IVPFSWA operator is defined as IVPFSWA: $\Psi^n \longrightarrow \Psi$

$$\text{IVPFSWA}\left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}\right) = \oplus_{j=1}^{m} \nu_j \left(\oplus_{i=1}^{n} \omega_i \mathcal{M}_{e_{ij}}\right)$$

Theorem 3.1

Let $\mathcal{M}_{e_{ij}} = \left(\left[\kappa^l_{ij}, \kappa^u_{ij}\right], \left[\delta^l_{ij}, \delta^u_{ij}\right]\right)$ be a collection of IVPFSNs, where $(i = 1, 2, 3 \ldots, n\ and\ j = 1, 2, 3, \ldots m)$, and the aggregated value is also an IVPFSN, such as

$$\text{IVPFSWA}\left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}\right)$$
$$= \left(\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa^l_{ij}, \kappa^u_{ij}\right]^2\right)^{\omega_i}\right)^{\nu_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\delta^l_{ij}, \delta^u_{ij}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

where ω_i and ν_j are weight vector for expert's and attributes respectively with given conditions $\omega_i > 0, \sum_{i=1}^{n} \omega_i = 1; \nu_j > 0, \sum_{j=1}^{m} \nu_j = 1$.

Proof. We shall prove the IVPFSWA operator by utilizing the principle of mathematical induction:

For $n = 1$, we get $\omega_1 = 1$. Then, we have

$$\text{IVPFSWA}\left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{1m}}\right) = \oplus_{j=1}^{m} \nu_j \mathcal{M}_{e_{1j}}$$

$$\text{IVPFSWA}\left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}\right) = \left(\sqrt{1-\prod_{j=1}^{m}\left(1-\left[\kappa^l_{1j}, \kappa^u_{1j}\right]^2\right)^{\nu_j}}, \prod_{j=1}^{m}\left(\left[\delta^l_{1j}, \delta^u_{1j}\right]\right)^{\nu_j}\right)$$

$$= \left(\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\left[\kappa^l_{ij}, \kappa^u_{ij}\right]^2\right)^{\omega_i}\right)^{\nu_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(\left[\delta^l_{ij}, \delta^u_{ij}\right]\right)^{\omega_i}\right)^{\nu_j}\right).$$

For $m=1$, we get $\nu_1=1$. Then, we have

$$\mathrm{IVPFSWA}\left(\mathcal{M}_{e_{11}},\mathcal{M}_{e_{21}},\ldots,\mathcal{M}_{e_{n1}}\right)=\oplus_{i=1}^{n}\omega_i\mathcal{M}_{e_{i1}}$$

$$=\left(\sqrt{1-\prod_{i=1}^{n}\left(1-\left[\kappa_{i1}^{l},\kappa_{i1}^{u}\right]^2\right)^{\omega_i}},\prod_{i=1}^{n}\left(\left[\delta_{i1}^{l},\delta_{1i}^{u}\right]\right)^{\omega_i}\right)$$

$$=\left(\sqrt{1-\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\left[\kappa_{ij}^{l},\kappa_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}},\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(\left[\delta_{ij}^{l},\delta_{ij}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

This shows that the above theorem holds for $n=1$ and $m=1$. Now, consider the above theorem also holds for $m=\alpha_1+1, n=\alpha_2$ and $m=\alpha_1, n=\alpha_2+1$, such as

$$\oplus_{j=1}^{\alpha_1+1}\nu_j\left(\oplus_{i=1}^{\alpha_2}\omega_i\mathcal{M}_{e_{ij}}\right)=\left(\sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left[\kappa_{ij}^{l},\kappa_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}},\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(\left[\delta_{ij}^{l},\delta_{ij}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

$$\oplus_{j=1}^{\alpha_1}\nu_j\left(\oplus_{i=1}^{\alpha_2+1}\omega_i\mathcal{M}_{e_{ij}}\right)=\left(\sqrt{1-\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(1-\left[\kappa_{ij}^{l},\kappa_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}},\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(\left[\delta_{ij}^{l},\delta_{ij}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

For $m=\alpha_1+1$ and $n=\alpha_2+1$, we have

$$\oplus_{j=1}^{\alpha_1+1}\nu_j\left(\oplus_{i=1}^{\alpha_2+1}\omega_i\mathcal{M}_{e_{ij}}\right)=\oplus_{j=1}^{\alpha_1+1}\nu_j\left(\oplus_{i=1}^{\alpha_2}\omega_i\mathcal{M}_{e_{ij}}\oplus\omega_{\alpha_2+1}\mathcal{M}_{e_{(\alpha_2+1)j}}\right)$$

$$=\oplus_{j=1}^{\alpha_1+1}\oplus_{i=1}^{\alpha_2}\nu_j\omega_i\mathcal{M}_{e_{ij}}\oplus_{j=1}^{\alpha_1+1}\nu_j\omega_{\alpha_2+1}\mathcal{M}_{e_{(\alpha_2+1)j}}$$

$$=\left(\sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1-\left[\kappa_{ij}^{l},\kappa_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\oplus\sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\left(1-\left[\kappa_{(\alpha_2+1)j}^{l},\kappa_{(\alpha_2+1)j}^{u}\right]^2\right)^{\omega_{\alpha_2+1}}\right)^{\nu_j}},\right.$$

$$\left.\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(\left[\delta_{ij}^{l},\delta_{ij}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}\oplus\prod_{j=1}^{\alpha_1+1}\left(\left(\left[\delta_{(\alpha_2+1)j}^{l},\delta_{(\alpha_2+1)j}^{u}\right]\right)^{\omega_{(\alpha_2+1)}}\right)^{\nu_j}\right)$$

$$=\left(\sqrt{1-\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_{2+1}}\left(1-\left[\kappa_{ij}^{l},\kappa_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}},\prod_{j=1}^{\alpha_{1+1}}\left(\prod_{i=1}^{\alpha_2+1}\left(\left[\delta_{ij}^{l},\delta_{ij}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

Therefore, it holds for $m=\alpha_1+1$ and $n=\alpha_2+1$. So, we can judge that the above theorem also holds for all values of m and n.

Example 3.1

Let $\chi = \{x_1, x_2, x_3\}$ be the set of specialists with weights $\omega_i = (0.38, 0.45, 0.17)^T$ who wants to choose a bike under some defined set of properties $\varphi = \{e_1 = \textit{Resale Value}, e_2 = \textit{Mileage}, e_3 = \textit{Cost of bike}\}$ with weights $\nu_j = (0.25, 0.45, 0.3)^T$. We suppose the rating values of the specialists for each property in the form of IVPFSNs $(\mathcal{M}, \varphi) = \left(\left[\kappa_{ij}^l, \kappa_{ij}^u\right], \left[\delta_{ij}^l, \delta_{ij}^u\right]\right)_{3\times 3}$ is given as

$$(\mathcal{M}, \varphi) = \begin{bmatrix} ([0.3, 0.8], [0.4, 0.5]) & ([0.4, 0.6], [0.3, 0.7]) & ([0.5, 0.8], [0.5, 0.6]) \\ ([0.1, 0.5], [0.2, 0.3]) & ([0.3, 0.8], [0.5, 0.7]) & ([0.2, 0.4], [0.2, 0.3]) \\ ([0.2, 0.9], [0.2, 0.3]) & ([0.5, 0.7], [0.2, 0.6]) & ([0.2, 0.4], [0.2, 0.8]) \end{bmatrix}$$

By using the above theorem, we have

$$\text{IVPFSWA}\left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{33}}\right) = \left(\sqrt{1 - \prod_{j=1}^{3}\left(\prod_{i=1}^{3}\left(1 - \left[\kappa_{ij}^l, \kappa_{ij}^u\right]^2\right)^{\omega_i}\right)^{\nu_j}}, \prod_{j=1}^{3}\left(\prod_{i=1}^{3}\left(\left[\delta_{ij}^l, \delta_{ij}^u\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\begin{array}{c} \left\{\begin{array}{c}[0.36, 0.91]^{0.38}\,[0.64, 0.84]^{0.45} \\ [0.36, 0.75]^{0.17}\end{array}\right\}^{0.17} \left\{\begin{array}{c}[0.75, 0.99]^{0.38} \\ [0.36, 0.91]^{0.45}\,[0.75, 0.99]^{0.17}\end{array}\right\}^{0.45} \\ \left\{\begin{array}{c}[0.19, 0.96]^{0.38}\,[0.51, 0.75]^{0.45} \\ [0.84, 0.96]^{0.17}\end{array}\right\}^{0.3}\end{array}\right)}, \\ \left(\begin{array}{c} \left\{\begin{array}{c}[0.4, 0.5]^{0.38}\,[0.3, 0.7]^{0.45} \\ [0.5, 0.6]^{0.17}\end{array}\right\}^{0.25} \left\{\begin{array}{c}[0.2, 0.7]^{0.38}\,[0.5, 0.7]^{0.45} \\ [0.2, 0.3]^{0.17}\end{array}\right\}^{0.45} \\ \left\{\begin{array}{c}[0.2, 0.3]^{0.38}\,[0.2, 0.6]^{0.45} \\ [0.2, 0.8]^{0.17}\end{array}\right\}^{0.3}\end{array}\right)\end{array}\right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\begin{array}{c} \left\{\begin{array}{c}[0.6783, 0.9648][0.8181, 0.9245] \\ [0.8406, 0.9523]\end{array}\right\}^{0.25} \left\{\begin{array}{c}[0.8964, 0.9962][0.6314, 0.9584] \\ [0.9523, 0.9983]\end{array}\right\}^{0.45} \\ \left\{\begin{array}{c}[0.5320, 0.9846][0.7386, 0.8786] \\ [0.9708, 0.9931]\end{array}\right\}^{0.3}\end{array}\right)}, \\ \left(\begin{array}{c} \left\{\begin{array}{c}[0.7060, 0.7486][0.5817, 0.8517] \\ [0.8888, 0.9168]\end{array}\right\}^{0.25} \left\{\begin{array}{c}[0.5425, 0.8732][0.7320, 0.8517] \\ [0.7606, 0.8149]\end{array}\right\}^{0.45} \\ \left\{\begin{array}{c}[0.5425, 0.6329][0.4847, 0.7946] \\ [0.7606, 0.9628]\end{array}\right\}^{0.3}\end{array}\right)\end{array}\right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\begin{array}{c}[0.4665, 0.8494]^{0.25}\,[0.5390, 0.9531]^{0.45} \\ [0.3815, 0.8591]^{0.3}\end{array}\right)}, \\ \left(\begin{array}{c}[0.3650, 0.5857]^{0.25}\,[0.3020, 0.6060]^{0.45} \\ [0.2000, 0.4842]^{0.3}\end{array}\right)\end{array}\right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\begin{array}{c}[0.8264, 0.9600][0.7572, 0.9786] \\ [0.7489, 0.9555]\end{array}\right)}, \\ \left(\begin{array}{c}[0.7773, 0.8748] \\ [0.5834, 0.7982][0.6170, 0.8045]\end{array}\right)\end{array}\right)$$

$$= \left(\sqrt{1-[0.7773, 0.8977]}, \atop [0.2798, 0.5617]\right)$$

$$= \left(\sqrt{[0.1023, 0.2227]}, [0.2798, 0.5617]\right)$$

$$= ([0.3198, 0.4719], [0.2798, 0.5617])$$

3.3 Properties of PFSWA Operator

3.3.1 Idempotency

If $\mathcal{M}_{e_{ij}} = \mathcal{M}_e = \left(\left[\kappa_{ij}^l, \kappa_{ij}^u\right], \left[\delta_{ij}^l, \delta_{ij}^u\right]\right) \forall i, j,$ then,

IVPFSWA $\left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}\right) = \mathcal{M}_e$

Proof: As we know that all $\mathcal{M}_{e_{ij}} = \mathcal{M}_e = \left(\left[\kappa_{ij}^l, \kappa_{ij}^u\right], \left[\delta_{ij}^l, \delta_{ij}^u\right]\right)$,then, we have

IVPFSWA $\left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}\right)$

$$= \left(\sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left[\kappa_{ij}^l, \kappa_{ij}^u\right]^2\right)^{\omega_i}\right)^{\nu_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\delta_{ij}^l, \delta_{ij}^u\right]\right)^{\omega_i}\right)^{\nu_j}\right)$$

$$= \left(\sqrt{1 - \left(\left(1 - \left[\kappa_{ij}^l, \kappa_{ij}^u\right]^2\right)^{\sum_{i=1}^{n}\omega_i}\right)^{\sum_{j=1}^{m}\nu_j}}, \left(\left(\left[\delta_{ij}^l, \delta_{ij}^u\right]\right)^{\sum_{i=1}^{n}\omega_i}\right)^{\sum_{j=1}^{m}\nu_j}\right)$$

As $\sum_{j=1}^{m}\nu_j = 1$ and $\sum_{i=1}^{n}\omega_i = 1$, then we have

$$= \left(\sqrt{1 - \left(1 - \left[\kappa_{ij}^l, \kappa_{ij}^u\right]^2\right)}, \left[\delta_{ij}^l, \delta_{ij}^u\right]\right)$$

$$= \left(\left[\kappa_{ij}^l, \kappa_{ij}^u\right], \left[\delta_{ij}^l, \delta_{ij}^u\right]\right)$$

$$= \mathcal{M}_e$$

Hence proved.

3.3.2 Boundedness

Let $\mathcal{M}_{e_{ij}}$ be a collection of PFSNs where $\mathcal{M}_{e_{ij}}^{-} = \left(\underset{j}{min}\ \underset{i}{min}\left\{\left[\kappa_{ij}^l, \kappa_{ij}^u\right]\right\}, \underset{j}{max}\ \underset{i}{max}\left\{\left[\delta_{ij}^l, \delta_{ij}^u\right]\right\}\right)$ and $\mathcal{M}_{e_{ij}}^{+} = \left(\underset{j}{max}\ \underset{i}{max}\left\{\left[\kappa_{ij}^l, \kappa_{ij}^u\right]\right\}, \underset{j}{min}\ \underset{i}{min}\left\{\left[\delta_{ij}^l, \delta_{ij}^u\right]\right\}\right)$, then

$\mathcal{M}_{e_{ij}}^{-} \leq$ IVPFSWA $\left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}\right) \leq \mathcal{M}_{e_{ij}}^{+}$

Proof. As we know that $\mathcal{M}_{e_{ij}} = \left\langle \left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right], \left[\delta_{ij}^{l}, \delta_{ij}^{u}\right] \right\rangle$ be an IVPFSN, then

$$\underset{j}{min}\ \underset{i}{min} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\} \leq \left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2 \leq \underset{j}{max}\ \underset{i}{max} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\}$$

$$\Rightarrow 1 - \underset{j}{max}\ \underset{i}{max} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\} \leq 1 - \left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2 \leq 1 - \underset{j}{min}\ \underset{i}{min} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\}$$

$$\Leftrightarrow \left(1 - \underset{j}{max}\ \underset{i}{max} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\}\right)^{\omega_i} \leq \left(1 - \left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right)^{\omega_i} \leq \left(1 - \underset{j}{min}\ \underset{i}{min} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\}\right)^{\omega_i}$$

$$\Leftrightarrow \left(1 - \underset{j}{max}\ \underset{i}{max} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\}\right)^{\sum_{i=1}^{n} \omega_i} \leq \prod_{i=1}^{n} \left(1 - \left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right)^{\omega_i} \leq \left(1 - \underset{j}{min}\ \underset{i}{min} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\}\right)^{\sum_{i=1}^{n} \omega_i}$$

$$\Leftrightarrow \left(1 - \underset{j}{max}\,\underset{i}{max} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\}\right)^{\sum_{j=1}^{n} \nu_j} \leq \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j} \leq \left(1 - \underset{j}{min}\,\underset{i}{min} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\}\right)^{\sum_{j=1}^{n} \nu_j}$$

$$\Leftrightarrow 1 - \underset{j}{max}\ \underset{i}{max} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\} \leq \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j} \leq 1 - \underset{j}{min}\ \underset{i}{min} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\}$$

$$\Leftrightarrow \underset{j}{min}\ \ \underset{i}{min} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\} \leq 1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j} \leq \underset{j}{max}\ \underset{i}{max} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right\}$$

$$\Leftrightarrow \underset{j}{min}\ \underset{i}{min} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]\right\} \leq \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}} \leq \underset{j}{max}\ \underset{i}{max} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]\right\} \tag{a}$$

Similarly, we can prove that

$$\underset{j}{min}\ \underset{i}{min} \left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right\} \leq \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right)^{\omega_i}\right)^{\nu_j} \leq \underset{j}{max}\ \underset{i}{max} \left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right\} \tag{b}$$

Let IVPFSWA $(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}) = \langle [\kappa_{\sigma}^{l}, \kappa_{\sigma}^{u}], [\delta_{\sigma}^{l}, \delta_{\sigma}^{u}] \rangle = \mathcal{M}_{\sigma}$, then inequalities (a) and (b) can be transferred into the form: $\underset{j}{min}\ \underset{i}{min} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]\right\} \leq \mathcal{M}_{\sigma} \leq \underset{j}{max}\ \underset{i}{max} \left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]\right\}$ and $\underset{j}{min}\ \underset{i}{min} \left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right\} \leq \mathcal{M}_{\sigma} \leq \underset{j}{max}\ \underset{i}{max} \left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right\}$, respectively.

So, by using the score function, we have

$$\mathbb{S}(\mathcal{M}_{\sigma})=\frac{\left(\kappa_{\sigma}^{l}\right)^{2}+\left(\kappa_{\sigma}^{u}\right)^{2}-\left(\delta_{\sigma}^{l}\right)^{2}-\left(\delta_{\sigma}^{u}\right)^{2}}{2}\leq \underset{j}{max}\ \underset{i}{max}\left\{\left[\kappa_{ij}^{l},\kappa_{ij}^{u}\right]\right\}-\underset{j}{min}\ \underset{i}{min}\left\{\left[\delta_{ij}^{l},\delta_{ij}^{u}\right]\right\}$$

$$=\mathbb{S}\left(\mathcal{M}_{e_{ij}}^{-}\right)$$

$$\mathbb{S}(\mathcal{M}_{\sigma})=\frac{\left(\kappa_{\sigma}^{l}\right)^{2}+\left(\kappa_{\sigma}^{u}\right)^{2}-\left(\delta_{\sigma}^{l}\right)^{2}-\left(\delta_{\sigma}^{u}\right)^{2}}{2}\geq \underset{j}{min}\ \underset{i}{min}\left\{\left[\kappa_{ij}^{l},\kappa_{ij}^{u}\right]\right\}-\underset{j}{max}\ \underset{i}{max}\left\{\left[\delta_{ij}^{l},\delta_{ij}^{u}\right]\right\}=\mathbb{S}\left(\mathcal{M}_{e_{ij}}^{+}\right)$$

Then, by order relation between two IVPFSNs, we have

$$\mathcal{M}_{e_{ij}}^{-}\leq \text{IVPFSWA}\ \left(\mathcal{M}_{e_{11}},\mathcal{M}_{e_{12}},\ldots,\mathcal{M}_{e_{nm}}\right)\leq \mathcal{M}_{e_{ij}}^{+}$$

Hence proved.

3.3.3 Shift Invariance

If $\mathcal{M}_{e}=\left\langle\left[\kappa^{l},\kappa^{u}\right],\left[\delta^{l},\delta^{u}\right]\right\rangle$ be an IVPFSN, then

$$\text{IVPFSWA}(\mathcal{M}_{e_{11}}\oplus\mathcal{M}_{e},\mathcal{M}_{e_{12}}\oplus\mathcal{M}_{e},\ldots,\mathcal{M}_{e_{nm}}\oplus\mathcal{M}_{e})=\text{IVPFSWA}\ (\mathcal{M}_{e_{11}},\mathcal{M}_{e_{12}},\ldots,\mathcal{M}_{e_{nm}})\oplus\mathcal{M}_{e}$$

Proof. Consider $\mathcal{M}_{e}$ and $\mathcal{M}_{e_{ij}}$ be two IVPFSNs. Then, by operational laws defined under IVPFSNs defined above, we have

$$\mathcal{M}_{e}\oplus\mathcal{M}_{e_{ij}}=\left(\sqrt{\left[\kappa^{l},\kappa^{u}\right]+\left[\kappa_{ij}^{l},\kappa_{ij}^{u}\right]^{2}-\left[\kappa^{l},\kappa^{u}\right]\left[\kappa_{ij}^{l},\kappa_{ij}^{u}\right]^{2}},\left[\delta^{l},\delta^{u}\right]\left[\delta_{ij}^{l},\delta_{ij}^{u}\right]\right),\quad \text{therefore}$$

$$\text{IVPFSWA}\ \left(\mathcal{M}_{e_{11}}\oplus\mathcal{M}_{e},\mathcal{M}_{e_{12}}\oplus\mathcal{M}_{e},\ldots,\mathcal{M}_{e_{nm}}\oplus\mathcal{M}_{e}\right)=\oplus_{j=1}^{m}\nu_{j}\left(\oplus_{i=1}^{n}\omega_{i}\left(\mathcal{M}_{e_{ij}}\oplus\mathcal{M}_{e}\right)\right)$$

$$=\left(\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa_{ij}^{l},\kappa_{ij}^{u}\right]^{2}\right)^{\omega_{i}}\left(1-\left[\kappa^{l},\kappa^{u}\right]^{2}\right)^{\omega_{i}}\right)^{\nu_{j}}},\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\delta_{ij}^{l},\delta_{ij}^{u}\right]\right)^{\omega_{i}}\left(\left[\delta^{l},\delta^{u}\right]\right)^{\omega_{i}}\right)^{\nu_{j}}\right)$$

$$=\left(\sqrt{1-\left(1-\left[\kappa^{l},\kappa^{u}\right]^{2}\right)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa_{ij}^{l},\kappa_{ij}^{u}\right]^{2}\right)^{\omega_{i}}\right)^{\nu_{j}}},\left[\delta^{l},\delta^{u}\right]\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\delta_{ij}^{l},\delta_{ij}^{u}\right]\right)^{\omega_{i}}\right)^{\nu_{j}}\right)$$

$$=\left(\left(\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa_{ij}^{l},\kappa_{ij}^{u}\right]^{2}\right)^{\omega_{i}}\right)^{\nu_{j}}},\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\delta_{ij}^{l},\delta_{ij}^{u}\right]\right)^{\omega_{i}}\right)^{\nu_{j}}\right)\oplus\left(\left[\kappa^{l},\kappa^{u}\right],\left[\delta^{l},\delta^{u}\right]\right)\right)$$

$$=\text{IVPFSWA}\ \left(\mathcal{M}_{e_{11}},\mathcal{M}_{e_{12}},\ldots,\mathcal{M}_{e_{nm}}\right)\oplus\mathcal{M}_{e}$$

Hence proved.

3.3.4 Homogeneity

Prove that IVPFSWA $(\beta\mathcal{M}_{e_{11}}, \beta\mathcal{M}_{e_{12}}, \ldots, \beta\mathcal{M}_{e_{nm}}) = \beta$ IVPFSWA $(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}})$ for any positive real number β.

Proof. Let $\mathcal{M}_{e_{ij}}$ be an IVPFSN and $\beta > 0$, then by using the operational laws mentioned above, we have

$$\beta\mathcal{M}_{e_{ij}} = \left(\sqrt{1-\left(1-\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^{2}\right)^{\beta}}, \left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{\beta}\right)$$

So,

$$\beta\mathcal{M}_{e_{11}}, \beta\mathcal{M}_{e_{12}}, \ldots, \beta\mathcal{M}_{e_{nm}})$$

$$= \left(\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^{2}\right)^{\beta\omega_i}\right)^{\nu_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right)^{\beta\omega_i}\right)^{\nu_j}\right)$$

$$= \left(\sqrt{1-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}\right)^{\beta}}, \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}\right)^{\beta}\right)$$

$$= \beta \text{ IVPFSWA } \left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}\right)$$

which completes the proof.

3.4 Interval Valued Pythagorean Fuzzy Soft Weighted Geometric Operator

Let $\mathcal{M}_{e_{ij}} = \left\langle\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right], \left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right\rangle$ be a collection of interval-valued Pythagorean fuzzy soft numbers (IVPFSNs), and ω_i and ν_j are the weight vector for experts and parameters, respectively, with given conditions $\omega_i > 0, \sum_{i=1}^{n}\omega_i = 1; \nu_j > 0, \sum_{j=1}^{m}\nu_j = 1$. Then, the IVPFSWG operator is defined as IVPFSWG: $\Psi^n \longrightarrow \Psi$

$$\text{IVPFSWG } \left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}\right) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\mathcal{M}_{e_{ij}}^{\omega_i}\right)^{\nu_j}$$

Theorem 3.2

Let $\mathcal{M}_{e_{ij}} = \left\langle\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right], \left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right\rangle$ be a collection of interval-valued Pythagorean fuzzy soft numbers (IVPFSNs). Then, the aggregated value obtained by using the IVPFSWG operator is also IVPFSN and

$$\text{IVPFSWG } \left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}\right)$$

$$= \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}, \sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}}\right)$$

where ω_i and ν_j are weight vector for expert's and attributes respectively with given conditions $\omega_i > 0, \sum_{i=1}^{n}\omega_i = 1; \nu_j > 0, \sum_{j=1}^{m}\nu_j = 1$.

Proof. We can prove the IVPFSWG operator by using the principle of mathematical induction as follows:

For $n = 1$, we get $\omega_1 = 1$. Then, we have

$$\mathrm{IVPFSWG}\left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{1m}}\right) = \otimes_{j=1}^{m} \mathcal{M}_{e_{1j}}^{\nu_j}$$

$$\mathrm{IVPFSWG}\ \left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}\right)$$

$$= \left(\prod_{j=1}^{m}\left(\left[\kappa_{1j}^{l}, \kappa_{1j}^{u}\right]\right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{m}\left(1 - \left[\delta_{1j}^{l}, \delta_{1j}^{u}\right]^2\right)^{\nu_j}}\right)$$

$$= \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1 - \left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\right).$$

For $m = 1$, we get $\nu_1 = 1$. Then, we have

$$\mathrm{IVPFSWG}\left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{21}}, \ldots, \mathcal{M}_{e_{n1}}\right) = \otimes_{i=1}^{n}\left(\mathcal{M}_{e_{i1}}\right)^{\omega_i}$$

$$= \left(\prod_{i=1}^{n}\left(\left[\kappa_{i1}^{l}, \kappa_{1i}^{u}\right]\right)^{\omega_i}, \sqrt{1 - \prod_{i=1}^{n}\left(1 - \left[\delta_{i1}^{l}, \delta_{i1}^{u}\right]^2\right)^{\omega_i}}\right)$$

$$= \left(\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1 - \left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\right)$$

This shows that the above theorem holds for $n = 1$ and $m = 1$. Now, consider the above theorem also holds for $m = \alpha_1 + 1, n = \alpha_2$ and $m = \alpha_1, n = \alpha_2 + 1$, such as

$$\otimes_{j=1}^{\alpha_1+1}\left(\otimes_{i=1}^{\alpha_2}\left(\mathcal{M}_{e_{ij}}\right)^{\omega_i}\right)^{\nu_j}$$

$$= \left(\prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]\right)^{\omega_i}, \sqrt{1 - \prod_{j=1}^{\alpha_1+1}\left(\prod_{i=1}^{\alpha_2}\left(1 - \left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\right)\right)$$

$$\otimes_{j=1}^{\alpha_1}\left(\otimes_{i=1}^{\alpha_2+1}\left(\mathcal{M}_{e_{ij}}\right)^{\omega_i}\right)^{\nu_j}$$

$$= \left(\prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(\left[\kappa_{ij}^{l}, \delta_{ij}^{u}\right]\right)^{\omega_i}\right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{\alpha_1}\left(\prod_{i=1}^{\alpha_2+1}\left(1 - \left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^2\right)^{\omega_i}\right)^{\nu_j}}\right)$$

For $m = \alpha_1 + 1$ and $n = \alpha_2 + 1$, we have

$$\otimes_{j=1}^{\alpha_1+1}\left(\otimes_{i=1}^{\alpha_2+1}\left(\mathcal{M}_{e_{ij}}\right)^{\omega_i}\right)^{\nu_j} = \otimes_{j=1}^{\alpha_1+1}\left(\otimes_{i=1}^{\alpha_2}\left(\mathcal{M}_{e_{ij}}\right)^{\omega_i} \otimes \left(\mathcal{M}_{e_{(\alpha_2+1)j}}\right)^{\omega_{\alpha_2+1}}\right)^{\nu_j}$$

$$= \otimes_{j=1}^{\alpha_1+1} \otimes_{i=1}^{\alpha_2} \left(\left(\mathcal{M}_{e_{ij}} \right)^{\omega_i} \right)^{\nu_j} \otimes_{j=1}^{\alpha_1+1} \left(\left(\mathcal{M}_{e_{(\alpha_2+1)j}} \right)^{\omega_{\alpha_2+1}} \right)^{\nu_j}$$

$$= \left(\prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(\left[\kappa_{ij}^l, \kappa_{ij}^u \right] \right)^{\omega_i} \right)^{\nu_j} \otimes \prod_{j=1}^{\alpha_1+1} \left(\left(\left[\kappa_{(\alpha_2+1)j}^l, \kappa_{(\alpha_2+1)j}^u \right] \right)^{\omega(\alpha_2+1)} \right)^{\nu_j}, \right.$$

$$\left. \sqrt{1 - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_2} \left(1 - \left[\delta_{ij}^l, \delta_{ij}^u \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \otimes \sqrt{1 - \prod_{j=1}^{\alpha_1+1} \left(\left(1 - \left[\delta_{(\alpha_2+1)j}^l, \delta_{(\alpha_2+1)j}^u \right]^2 \right)^{\omega_{\alpha_2+1}} \right)^{\nu_j}} \right)$$

$$= \left(\prod_{j=1}^{\alpha_{1+1}} \left(\prod_{i=1}^{\alpha_2+1} \left(\left[\kappa_{ij}^l, \kappa_{ij}^u \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{\alpha_1+1} \left(\prod_{i=1}^{\alpha_{2+1}} \left(1 - \left[\delta_{ij}^l, \delta_{ij}^u \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)$$

It is clarified from the above equation that the theorem holds for $m = \alpha_1 + 1$ and $n = \alpha_2 + 1$. So, we can say that the theorem holds for all values of m and n.

Example 3.2

Again, consider Example 3.1 with rating values of the specialists for each property in the form of IVPFSNs $(\mathcal{M}, \varphi) = \left(\left[\kappa_{ij}^l, \kappa_{ij}^u \right], \left[\delta_{ij}^l, \delta_{ij}^u \right] \right)_{3\times 3}$ is given as

$$(\mathcal{M}, \varphi) = \begin{bmatrix} ([0.4, 0.5], [0.3, 0.8]) & ([0.3, 0.7][0.4, 0.6]) & ([0.5, 0.6], [0.5, 0.8]) \\ ([0.2, 0.3], [0.1, 0.5]) & ([0.5, 0.7], [0.3, 0.8]) & ([0.2, 0.3], [0.2, 0.4]) \\ ([0.2, 0.3], [0.2, 0.9]) & ([0.2, 0.6], [0.5, 0.7]) & ([0.2, 0.8], [0.2, 0.4]) \end{bmatrix}$$

By using the above theorem, we have

$\mathrm{IVPFSWG}\left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{33}} \right)$

$$= \left(\prod_{j=1}^{3} \left(\prod_{i=1}^{3} \left(\left[\kappa_{ij}^l, \kappa_{ij}^u \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{3} \left(\prod_{i=1}^{3} \left(1 - \left[\delta_{ij}^l, \delta_{ij}^u \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left\{ \begin{array}{c} [0.4, 0.5]^{0.38} [0.3, 0.7]^{0.45} \\ [0.5, 0.6]^{0.17} \end{array} \right\}^{0.25} \left\{ \begin{array}{c} [0.2, 0.7]^{0.38} [0.5, 0.7]^{0.45} \\ [0.2, 0.3]^{0.17} \end{array} \right\}^{0.45} \\ \left\{ \begin{array}{c} [0.2, 0.3]^{0.38} [0.2, 0.6]^{0.45} \\ [0.2, 0.8]^{0.17} \end{array} \right\}^{0.3} \end{array} \right), \\ \sqrt{1 - \left(\begin{array}{c} \left\{ \begin{array}{c} [0.36, 0.91]^{0.38} [0.64, 0.84]^{0.45} \\ [0.36, 0.75]^{0.17} \end{array} \right\}^{0.17} \left\{ \begin{array}{c} [0.75, 0.99]^{0.38} \\ [0.36, 0.91]^{0.45} [0.75, 0.99]^{0.17} \end{array} \right\}^{0.45} \\ \left\{ \begin{array}{c} [0.19, 0.96]^{0.38} [0.51, 0.75]^{0.45} \\ [0.84, 0.96]^{0.17} \end{array} \right\}^{0.3} \end{array} \right)} \end{array} \right)$$

$$= \left(\frac{\left(\begin{Bmatrix} [0.7060, 0.7486][0.5817, 0.8517] \\ [0.8888, 0.9168] \end{Bmatrix}^{0.25} \begin{Bmatrix} [0.5425, 0.8732][0.7320, 0.8517] \\ [0.7606, 0.8149] \end{Bmatrix}^{0.45} \\ \begin{Bmatrix} [0.5425, 0.6329][0.4847, 0.7946] \\ [0.7606, 0.9628] \end{Bmatrix}^{0.3} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.6783, 0.9648][0.8181, 0.9245] \\ [0.8406, 0.9523] \end{Bmatrix}^{0.25} \begin{Bmatrix} [0.8964, 0.9962][0.6314, 0.9584] \\ [0.9523, 0.9983] \end{Bmatrix}^{0.45} \\ \begin{Bmatrix} [0.5320, 0.9846][0.7386, 0.8786] \\ [0.9708, 0.9931] \end{Bmatrix}^{0.3} \right)}} \right)$$

$$= \left(\frac{\left([0.3650, 0.5857]^{0.25} [0.3020, 0.6060]^{0.45} \\ [0.2000, 0.4842]^{0.3} \right),}{\sqrt{1 - \left([0.4665, 0.8494]^{0.25} [0.5390, 0.9531]^{0.45} \\ [0.3815, 0.8591]^{0.3} \right)}} \right)$$

$$= \left(\frac{\left([0.7773, 0.8748] \\ [0.5834, 0.7982][0.6170, 0.8045] \right)}{\sqrt{1 - \left([0.8264, 0.9600][0.7572, 0.9786] \\ [0.7489, 0.9555] \right)}} \right)$$

$$= \left(\frac{[0.2798, 0.5617],}{\sqrt{1 - [0.7773, 0.8977]}} \right)$$

$$= \left([0.2798, 0.5617], \sqrt{[0.1023, 0.2227]} \right)$$

$$= ([0.2798, 0.5617], [0.3198, 0.4719])$$

3.5 Properties of IVPFSWG

3.5.1 Idempotency

If $\boldsymbol{\mathcal{M}}_{e_{ij}} = \boldsymbol{\mathcal{M}}_e = \left(\left[\kappa_{ij}^l, \kappa_{ij}^u \right], \left[\delta_{ij}^l, \delta_{ij}^u \right] \right) \forall i, j$, then,

IVPFSWG $\left(\boldsymbol{\mathcal{M}}_{e_{11}}, \boldsymbol{\mathcal{M}}_{e_{12}}, \ldots, \boldsymbol{\mathcal{M}}_{e_{nm}} \right) = \boldsymbol{\mathcal{M}}_e$

Proof. As we know that all $\boldsymbol{\mathcal{M}}_{e_{ij}} = \boldsymbol{\mathcal{M}}_e = \left(\left[\kappa_{ij}^l, \kappa_{ij}^u \right], \left[\delta_{ij}^l, \delta_{ij}^u \right] \right)$, then, we have

IVPFSWG $\left(\boldsymbol{\mathcal{M}}_{e_{11}}, \boldsymbol{\mathcal{M}}_{e_{12}}, \ldots, \boldsymbol{\mathcal{M}}_{e_{nm}} \right)$

$$= \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\kappa_{ij}^l, \kappa_{ij}^u \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\delta_{ij}^l, \delta_{ij}^u \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)$$

$$= \left(\left(\left(\left[\kappa_{ij}^l, \kappa_{ij}^u \right] \right)^{\sum_{i=1}^{n} \omega_i} \right)^{\sum_{j=1}^{m} \nu_j}, \sqrt{1 - \left(\left(1 - \left[\delta_{ij}^l, \delta_{ij}^u \right]^2 \right)^{\sum_{i=1}^{n} \omega_i} \right)^{\sum_{j=1}^{m} \nu_j}} \right)$$

As $\sum_{j=1}^{m} \nu_j = 1$ and $\sum_{i=1}^{n} \omega_i = 1$, then we have

$$= \left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right], \sqrt{1-\left(1-\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right)}\right)$$

$$= \left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right], \left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right)$$

$$= \mathcal{M}_{e}$$

Hence proved.

3.5.2 Boundedness

Let $\mathcal{M}_{e_{ij}}$ be a collection of PFSNs where $\mathcal{M}_{e_{ij}}^{-} = \left(\underset{j}{min}\ \underset{i}{min}\left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]\right\}, \underset{j}{max}\ \underset{i}{max}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right\}\right)$ and $\mathcal{M}_{e_{ij}}^{+} = \left(\underset{j}{max}\ \underset{i}{max}\left\{\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right]\right\}, \underset{j}{min}\ \underset{i}{min}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right\}\right)$, then

$$\mathcal{M}_{e_{ij}}^{-} \leq \text{IVPFSWG}\ \left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}\right) \leq \mathcal{M}_{e_{ij}}^{+}$$

Proof. As we know that $\mathcal{M}_{e_{ij}} = \left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u}\right], \left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]\right)$ be an IVPFSN, then

$$\underset{j}{min}\ \underset{i}{min}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\} \leq \left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2} \leq \underset{j}{max}\ \underset{i}{max}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\}$$

$$\Rightarrow 1-\underset{j}{max}\ \underset{i}{max}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\} \leq 1-\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2} \leq 1-\underset{j}{min}\ \underset{i}{min}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\}$$

$$\Leftrightarrow \left(1-\underset{j}{max}\ \underset{i}{max}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\}\right)^{\omega_i} \leq \left(1-\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right)^{\omega_i} \leq \left(1-\underset{j}{min}\ \underset{i}{min}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\}\right)^{\omega_i}$$

$$\Leftrightarrow \left(1-\underset{j}{max}\ \underset{i}{max}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\}\right)^{\sum_{i=1}^{n}\omega_i} \leq \prod_{i=1}^{n}\left(1-\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right)^{\omega_i} \leq \left(1-\underset{j}{min}\ \underset{i}{min}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\}\right)^{\sum_{i=1}^{n}\omega_i}$$

$$\Leftrightarrow \left(1-\underset{j}{max}\ \underset{i}{max}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\}\right)^{\sum_{j=1}^{n}\nu_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right)^{\omega_i}\right)^{\nu_j}$$

$$\leq \left(1-\underset{j}{min}\ \underset{i}{min}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\}\right)^{\sum_{j=1}^{n}\nu_j}$$

$$\Leftrightarrow 1-\underset{j}{max}\ \underset{i}{max}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right)^{\omega_i}\right)^{\nu_j} \leq 1-\underset{j}{min}\ \underset{i}{min}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\}$$

$$\Leftrightarrow \underset{j}{min}\ \underset{i}{min}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\} \leq 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right)^{\omega_i}\right)^{\nu_j} \leq \underset{j}{max}\ \underset{i}{max}\left\{\left[\delta_{ij}^{l}, \delta_{ij}^{u}\right]^{2}\right\}$$

$$\Leftrightarrow \min_j \min_i \left\{\left[\delta_{ij}^l, \delta_{ij}^u\right]\right\} \leq \sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left[\delta_{ij}^l, \delta_{ij}^u\right]^2\right)^{\omega_i}\right)^{\nu_j}} \leq \max_j \max_i \left\{\left[\delta_{ij}^l, \delta_{ij}^u\right]\right\} \tag{c}$$

Similarly, we can prove that

$$\min_j \min_i \left\{\left[\kappa_{ij}^l, \kappa_{ij}^u\right]\right\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left[\kappa_{ij}^l, \kappa_{ij}^u\right]\right)^{\omega_i}\right)^{\nu_j} \leq \max_j \max_i \left\{\left[\kappa_{ij}^l, \kappa_{ij}^u\right]\right\} \tag{d}$$

Let IVPFSWG $(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}) = \langle[\kappa_\sigma^l, \kappa_\sigma^u], [\delta_\sigma^l, \delta_\sigma^u]\rangle = \mathcal{M}_\sigma$, then inequalities (c) and (d) can be transferred into the form:

$\min_j \min_i \left\{\left[\kappa_{ij}^l, \kappa_{ij}^u\right]\right\} \leq \mathcal{M}_\sigma \leq \max_j \max_i \left\{\left[\kappa_{ij}^l, \kappa_{ij}^u\right]\right\}$ and $\min_j \min_i \left\{\left[\delta_{ij}^l, \delta_{ij}^u\right]\right\} \leq \mathcal{M}_\sigma \leq \max_j \max_i \left\{\left[\delta_{ij}^l, \delta_{ij}^u\right]\right\}$, respectively.

So, by using the score function, we have

$$\mathbb{S}(\mathcal{M}_\sigma) = \frac{(\kappa_\sigma^l)^2 + (\kappa_\sigma^u)^2 - (\delta_\sigma^l)^2 - (\delta_\sigma^u)^2}{2} \leq \max_j \max_i \left\{\left[\kappa_{ij}^l, \kappa_{ij}^u\right]\right\} - \min_j \min_i \left\{\left[\delta_{ij}^l, \delta_{ij}^u\right]\right\} = \mathbb{S}\left(\mathcal{M}_{e_{ij}}^-\right)$$

$$\mathbb{S}(\mathcal{M}_\sigma) = \frac{(\kappa_\sigma^l)^2 + (\kappa_\sigma^u)^2 - (\delta_\sigma^l)^2 - (\delta_\sigma^u)^2}{2} \geq \min_j \min_i \left\{\left[\kappa_{ij}^l, \kappa_{ij}^u\right]\right\} - \max_j \max_i \left\{\left[\delta_{ij}^l, \delta_{ij}^u\right]\right\} = \mathbb{S}\left(\mathcal{M}_{e_{ij}}^+\right)$$

Then, by order relation between two IVPFSNs, we have

$\mathcal{M}_{e_{ij}}^- \leq \text{IVPFSWG}\ \left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}\right) \leq \mathcal{M}_{e_{ij}}^+$

Hence proved.

3.5.3 Shift Invariance

If $\mathcal{M}_e = \left(\left[\kappa^l, \kappa^u\right], \left[\delta^l, \delta^u\right]\right)$ be an IVPFSN, then

IVPFSWG $(\mathcal{M}_{e_{11}} \oplus \mathcal{M}_e, \mathcal{M}_{e_{12}} \oplus \mathcal{M}_e, \ldots, \mathcal{M}_{e_{nm}} \oplus \mathcal{M}_e) =$ IVPFSW$(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}}) \oplus \mathcal{M}_e$

Proof. Consider $\mathcal{M}_e$ and $\mathcal{M}_{e_{ij}}$ be two IVPFSNs. Then, by operational laws defined under IVPFSNs defined above, we have

$\mathcal{M}_e \oplus \mathcal{M}_{e_{ij}} = \left(\sqrt{\left[\kappa^l, \kappa^u\right]^2 + \left[\kappa_{ij}^l, \kappa_{ij}^u\right]^2 - \left[\kappa^l, \kappa^u\right]^2\left[\kappa_{ij}^l, \kappa_{ij}^u\right]^2}, \left[\delta^l, \delta^u\right]\left[\delta_{ij}^l, \delta_{ij}^u\right]\right)$, therefore

IVPFSWG $(\mathcal{M}_{e_{11}} \oplus \mathcal{M}_e, \mathcal{M}_{e_{12}} \oplus \mathcal{M}_e, \ldots, \mathcal{M}_{e_{nm}} \oplus \mathcal{M}_e)$

$= \oplus_{j=1}^m \nu_j \left(\oplus_{i=1}^n \omega_i \left(\mathcal{M}_{e_{ij}} \oplus \mathcal{M}_e\right)\right)$

$$= \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right] \right)^{\omega_i} \left(\left[\kappa^{l}, \kappa^{u} \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\delta_{ij}^{l}, \delta_{ij}^{u} \right]^2 \right)^{\omega_i} \left(1 - \left[\delta^{l}, \delta^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)$$

$$= \left(\left[\kappa^{l}, \kappa^{u} \right] \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \left(1 - \left[\delta^{l}, \delta^{u} \right]^2 \right) \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\delta_{ij}^{l}, \delta_{ij}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)$$

$$= \left(\left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\delta_{ij}^{l}, \delta_{ij}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right) \oplus \left(\left[\kappa^{l}, \kappa^{u} \right], \left[\delta^{l}, \delta^{u} \right] \right) \right)$$

$$= \text{IVPFSWG } \left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}} \right) \oplus \mathcal{M}_{e}$$

Hence proved.

3.5.4 Homogeneity

Prove that IVPFSWG $(\beta\mathcal{M}_{e_{11}}, \beta\mathcal{M}_{e_{12}}, \ldots, \beta\mathcal{M}_{e_{nm}}) = \beta$ IVPFSWA $(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}})$ for any positive real number β.

Proof: Let $\mathcal{M}_{e_{ij}}$ be an IVPFSN and $\beta > 0$, then by using the operational laws mentioned above, we have

$$\beta\mathcal{M}_{e_{ij}} = \left(\sqrt{1 - \left(1 - \left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right]^2 \right)^{\beta}}, \left[\delta_{ij}^{l}, \delta_{ij}^{u} \right]^{\beta} \right)$$

So,

$$\text{IVPFSWG } \left(\beta\mathcal{M}_{e_{11}}, \beta\mathcal{M}_{e_{12}}, \ldots, \beta\mathcal{M}_{e_{nm}} \right)$$

$$= \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right] \right)^{\beta\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\delta_{ij}^{l}, \delta_{ij}^{u} \right]^2 \right)^{\beta\omega_i} \right)^{\nu_j}} \right)$$

$$= \left(\left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right] \right)^{\omega_i} \right)^{\nu_j} \right)^{\beta}, \sqrt{1 - \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left[\delta_{ij}^{l}, \delta_{ij}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j} \right)^{\beta}} \right)$$

$$= \beta \, \text{IVPFSWG } \left(\mathcal{M}_{e_{11}}, \mathcal{M}_{e_{12}}, \ldots, \mathcal{M}_{e_{nm}} \right)$$

which completes the proof.

4 Multi-Attribute Group Decision-Making Approach Based on Proposed Operators

In this section, a decision-making (DM) approach for solving multi-attribute group decision-making (MAGDM) problems based on proposed IVPFSWA and IVPFSWG operators has been developed along with numerical examples.

4.1 Proposed Approach

Let $\mathfrak{I} = \{\mathfrak{I}^1, \mathfrak{I}^2, \mathfrak{I}^3, \ldots, \mathfrak{I}^s\}$ be the set of s alternatives, $X = \{x_1, x_2, x_3, \ldots x_r\}$ be the set of r specialists (decision-makers) and $\varphi = \{e_1, e_2, e_3, \ldots, e_m\}$ be the set of m attributes. Let the weighted vector of experts $X_i\,(i = 1, 2, 3, \ldots, r)$ is $\omega_i = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ such that $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$ and the weighted vector of attributes $e_i\,(i = 1, 2, 3, \ldots, m)$ is $\nu_j = (\nu_1, \nu_2, \nu_3, \ldots, \nu_n)^T$ such that $\nu_j > 0, \sum_{j=1}^{n} \nu_j = 1$. A team of specialists provides the decision matrix in the form of IVPFSNs such as $D_{m\times n}\left(\boldsymbol{\mathcal{M}}_{e_{ij}}\right) = \left(\left[\kappa_{ij}^l, \kappa_{ij}^u\right], \left[\delta_{ij}^l, \delta_{ij}^u\right]\right)_{m\times n}$.

The procedure to apply proposed IVPFSWG and IVPFSWA operators for solving the MAGDM problem is summarized in the following steps:

Step-1: Obtain a decision matrix in the form of PFSNs for alternatives relative to experts.

$$D_{m\times n}\left(\boldsymbol{\mathcal{M}}_{e_{ij}}\right) = \left(\left[\kappa_{ij}^l, \kappa_{ij}^u\right], \left[\delta_{ij}^l, \delta_{ij}^u\right]\right)_{m\times n}$$

$$= \begin{bmatrix} \left(\left[\kappa_{11}^l, \kappa_{11}^u\right], \left[\delta_{11}^l, \delta_{11}^u\right]\right) & \left(\left[\kappa_{12}^l, \kappa_{12}^u\right], \left[\delta_{12}^l, \delta_{122}^u\right]\right) & \cdots & \left(\left[\kappa_{1n}^l, \kappa_{1n}^u\right], \left[\delta_{1n}^l, \delta_{1n}^u\right]\right) \\ \left(\left[\kappa_{21}^l, \kappa_{21}^u\right], \left[\delta_{21}^l, \delta_{21}^u\right]\right) & \left(\left[\kappa_{22}^l, \kappa_{22}^u\right], \left[\delta_{22}^l, \delta_{22}^u\right]\right) & \cdots & \left(\left[\kappa_{2n}^l, \kappa_{2n}^u\right], \left[\delta_{2n}^l, \delta_{2n}^u\right]\right) \\ \vdots & & \ddots & \vdots \\ \left(\left[\kappa_{m1}^l, \kappa_{m1}^u\right], \left[\delta_{m1}^l, \delta_{m1}^u\right]\right) & \left(\left[\kappa_{m2}^l, \kappa_{m2}^u\right], \left[\delta_{m2}^l, \delta_{m2}^u\right]\right) & \cdots & \left(\left[\kappa_{mn}^l, \kappa_{mn}^u\right], \left[\delta_{mn}^l, \delta_{mn}^u\right]\right) \end{bmatrix}, \text{ where}$$

$0 \le \kappa_{ij}^l, \kappa_{ij}^u, \delta_{ij}^l, \delta_{ij}^u \le 1$ And $0 \le \left(\kappa_{ij}^u\right)^2 + \left(\delta_{ij}^u\right)^2 \le 1 \forall i, j$ are given in Tables 1–4.

Step-2: By using the normalization formula, normalize the decision matrix to convert the rating value of cost type parameters into benefit type parameters.

$$M_{e_{ij}} = \begin{cases} \boldsymbol{\mathcal{M}}_{e_{ij}}^c = \left(\left[\kappa_{ij}^l, \kappa_{ij}^u\right], \left[\delta_{ij}^l, \delta_{ij}^u\right]\right)_{n\times m} & \textit{cost type parameter} \\ \boldsymbol{\mathcal{M}}_{e_{ij}} = \left(\left[\delta_{ij}^l, \delta_{ij}^u\right], \left[\kappa_{ij}^l, \kappa_{ij}^u\right]\right)_{n\times m} & \textit{benefit type parameter} \end{cases}$$

Step-3: Use the developed IVPFSWG and IVPFSWA operators to aggregate the IVPFSNs $\boldsymbol{\mathcal{M}}_{e_{ij}}$ for each alternative $\mathfrak{I} = \{\mathfrak{I}^1, \mathfrak{I}^2, \mathfrak{I}^3, \ldots, \mathfrak{I}^s\}$ into the decision matrix $\boldsymbol{\mathcal{M}}_{ij}$.

Step-4: Calculate the score values of $\boldsymbol{\mathcal{M}}$ for all alternatives.

Step-5: Select the alternative having maximum score value and examine the ranking.

4.2 Numerical Example

Suppose a person wants to buy a car and he has four alternatives such as $\mathfrak{I}^1$, $\mathfrak{I}^2$, $\mathfrak{I}^3$ and $\mathfrak{I}^4$. There are four considered attributes according to which the person has to take the decision such as e_1; price of the car, e_2; comfortability, e_3; resale value, and, e_4; growth rate with the weighted vector$\nu = (0.3, 0.1, 0.2, 0.4)^T$. Here e_1, e_3 are cost type parameters and e_2, e_4 are benefit type parameters. The person hires a team of four experts $X_r(r = 1, 2, 3, 4)$ for decision making with the weighted vector $\omega = (0.1, 0.2, 0.4, 0.3)^T$.

4.2.1 By IVPFSWA Operator

Step-1: Obtain Pythagorean fuzzy soft decision matrices (Tables 1–4).

Table 1: IVPFS decision matrix for $\mathfrak{I}^1$

	e_1	e_2	e_3	e_4
x_1	$([.4, .5], [.2, .5])$	$([.7, .8], [.5, .6])$	$([.4, .6], [.2, .5])$	$([.2, .4], [.2, .6])$
x_2	$([.2, .7], [.2, .6])$	$([.1, .6], [.4, .5])$	$([.2, .3], [.4, .8])$	$([.2, .5], [.4, .7])$
x_3	$([.3, .5], [.1, .4])$	$([.4, .6], [.2, .7])$	$([.4, .7], [.3, .7])$	$([.5, .7], [.2, .4])$
x_4	$([.4, .6], [.3, .7])$	$([.4, .5], [.3, .7])$	$([.3, .6], [.3, .5])$	$([.3, .6], [.3, .5])$

Table 2: IVPFS decision matrix for $\mathfrak{I}^2$

	e_1	e_2	e_3	e_4
x_1	$([.3, .6], [.5, .6])$	$([.2, .7], [.5, .7])$	$([.2, .7], [.4, .5])$	$([.6, .7], [.5, .8])$
x_2	$([.3, .5], [.5, .8])$	$([.1, .4], [.4, .5])$	$([.1, .5], [.3, .7])$	$([.4, .5], [.3, .6])$
x_3	$([.2, .6], [.1, .4])$	$([.1, .2], [.2, .9])$	$([.4, .7], [.3, .8])$	$([.5, .8], [.2, .6])$
x_4	$([.2, .3], [.3, .8])$	$([.3, .5], [.2, .8])$	$([.3, .7], [.2, .6])$	$([.1, .7], [.3, .6])$

Table 3: IVPFS decision matrix for $\mathfrak{I}^3$

	e_1	e_2	e_3	e_4
x_1	$([.3, .4], [.2, .7])$	$([.3, .4], [.4, .6])$	$([.5, .6], [.4, .5])$	$([.3, .4], [.3, .6])$
x_2	$([.4, .6], [.3, .7])$	$([.3, .5], [.2, .3])$	$([.3, .5], [.5, .8])$	$([.2, .6], [.2, .4])$
x_3	$([.2, .4], [.3, .4])$	$([.3, .5], [.3, .7])$	$([.3, .7], [.3, .8])$	$([.1, .3], [.5, .6])$
x_4	$([.3, .7], [.3, .7])$	$([.3, .5], [.2, .4])$	$([.2, .5], [.3, .6])$	$([.3, .4], [.3, .7])$

Table 4: IVPFS decision matrix for $\mathfrak{I}^4$

	e_1	e_2	e_3	e_4
x_1	$([.3, .5], [.2, .6])$	$([.2, .6], [.4, .7])$	$([.2, .5], [.3, .6])$	$([.5, .7], [.6, .8])$
x_2	$([.2, .7], [.3, .8])$	$([.1, .5], [.4, .7])$	$([.5, .7], [.4, .5])$	$([.2, .5], [.3, .4])$
x_3	$([.2, .5], [.1, .6])$	$([.2, .5], [.1, .5])$	$([.2, .4], [.2, .7])$	$([.3, .5], [.1, .5])$
x_4	$([.2, .4], [.5, .8])$	$([.2, .5], [.5, .8])$	$([.2, .7], [.3, .6])$	$([.2, .5], [.4, .5])$

Step-2: Because e_1, e_3 are cost type parameters, so utilized the normalization formula to obtain normalized Pythagorean fuzzy soft decision matrices are given in the following Tables 5–8.

Table 5: Normalized IVPFS decision matrix for $\mathfrak{I}^1$

	e_1	e_2	e_3	e_4
x_1	$([.4,.5],[.2,.5])$	$([.5,.6],[.7,.8])$	$([.4,.6],[.2,.5])$	$([.2,.4],[.2,.6])$
x_2	$([.2,.7],[.2,.6])$	$([.4,.5],[.1,.6])$	$([.2,.3],[.4,.8])$	$([.2,.5],[.4,.7])$
x_3	$([.3,.5],[.1,.4])$	$([.2,.7],[.4,.6])$	$([.4,.7],[.3,.7])$	$([.5,.7],[.2,.4])$
x_4	$([.4,.6],[.3,.7])$	$([.3,.7],[.4,.5])$	$([.3,.6],[.3,.5])$	$([.3,.6],[.3,.5])$

Table 6: Normalized IVPFS decision matrix for $\mathfrak{I}^2$

	e_1	e_2	e_3	e_4
x_1	$([.3,.6],[.5,.6])$	$([.5,.7],[.2,.7])$	$([.2,.7],[.4,.5])$	$([.6,.7],[.5,.8])$
x_2	$([.3,.5],[.5,.8])$	$([.4,.5],[.1,.4])$	$([.1,.5],[.3,.7])$	$([.4,.5],[.3,.6])$
x_3	$([.2,.6],[.1,.4])$	$([.2,.9],[.1,.2])$	$([.4,.7],[.3,.8])$	$([.5,.8],[.2,.6])$
x_4	$([.2,.3],[.3,.8])$	$([.2,.8],[.3,.5])$	$([.3,.7],[.2,.6])$	$([.1,.7],[.3,.6])$

Table 7: Normalized IVPFS decision matrix for $\mathfrak{I}^3$

	e_1	e_2	e_3	e_4
x_1	$([.3,.4],[.2,.7])$	$([.4,.6],[.3,.4])$	$([.5,.6],[.4,.5])$	$([.3,.4],[.3,.6])$
x_2	$([.4,.6],[.3,.7])$	$([.2,.3],[.3,.5])$	$([.3,.5],[.5,.8])$	$([.2,.6],[.2,.4])$
x_3	$([.2,.4],[.3,.4])$	$([.3,.7],[.3,.5])$	$([.3,.7],[.3,.8])$	$([.1,.3],[.5,.6])$
x_4	$([.3,.7],[.3,.7])$	$([.2,.4],[.3,.5])$	$([.2,.5],[.3,.6])$	$([.3,.4],[.3,.7])$

Table 8: Normalized IVPFS decision matrix for $\mathfrak{I}^4$

	e_1	e_2	e_3	e_4
x_1	$([.3,.5],[.2,.6])$	$([.4,.7],[.2,.6])$	$([.2,.5],[.3,.6])$	$([.5,.7],[.6,.8])$
x_2	$([.2,.7],[.3,.8])$	$([.4,.7],[.1,.5])$	$([.5,.7],[.4,.5])$	$([.2,.5],[.3,.4])$
x_3	$([.2,.5],[.1,.6])$	$([.1,.5],[.2,.5])$	$([.2,.4],[.2,.7])$	$([.3,.5],[.1,.5])$
x_4	$([.2,.4],[.5,.8])$	$([.5,.8],[.2,.5])$	$([.2,.7],[.3,.6])$	$([.2,.5],[.4,.5])$

Step-3: Apply the proposed IVPFSWA operator on the acquired data, we will obtain an opinion of the decision-makers.

$$\Theta_1 = \left(\sqrt{1 - \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(1 - \left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}}, \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(\left[\delta_{ij}^{l}, \delta_{ij}^{u} \right] \right)^{\omega_i} \right)^{\nu_j} \right)$$

$$= \left(\sqrt{1 - \begin{pmatrix} \begin{Bmatrix} [0.75, 0.84]^{0.1} [0.64, 0.75]^{0.2} \\ [0.64, 0.84]^{0.4} [0.84, 0.96]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.51, 0.96]^{0.1} [0.75, 0.84]^{0.2} \\ [0.91, 0.96]^{0.4} [0.75, 0.96]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.75, 0.91]^{0.1} [0.51, 0.96]^{0.2} \\ [0.51, 0.84]^{0.4} [0.75, 0.91]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.64, 0.84]^{0.1} [0.51, 0.91]^{0.2} \\ [0.64, 0.91]^{0.4} [0.64, 0.91]^{0.3} \end{Bmatrix}^{0.4} \end{pmatrix}}, \begin{pmatrix} \begin{Bmatrix} [0.2, 0.5]^{0.1} [0.7, 0.8]^{0.2} \\ [0.2, 0.5]^{0.4} [0.2, 0.6]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.2, 0.6]^{0.1} [0.1, 0.6]^{0.2} \\ [0.4, 0.8]^{0.4} [0.4, 0.7]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.1, 0.4]^{0.1} [0.4, 0.6]^{0.2} \\ [0.3, 0.7]^{0.4} [0.2, 0.4]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.3, 0.7]^{0.1} [0.4, 0.5]^{0.2} \\ [0.3, 0.5]^{0.4} [0.3, 0.5]^{0.3} \end{Bmatrix}^{0.4} \end{pmatrix} \right)$$

$$= \left(\sqrt{1 - \begin{pmatrix} \begin{Bmatrix} [0.9716, 0.9827][0.9146, 0.9441] \\ [0.8365, 0.9326][0.9490, 0.9878] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.9349, 0.9949][0.9441, 0.9657] \\ [0.9629, 0.9838][0.9173, 0.9876] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.9716, 0.9906][0.8740, 0.9919] \\ [0.7639, 0.9326][0.9173, 0.9721] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.9564, 0.9827][0.8740, 0.9813] \\ [0.8365, 0.9630][0.8747, 0.9721] \end{Bmatrix}^{0.4} \end{pmatrix}}, \begin{pmatrix} \begin{Bmatrix} [0.8513, 0.9330][0.9311, 0.9564] \\ [0.5253, 0.7579][0.6170, 0.8579] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.8513, 0.9502][0.6310, 0.9029] \\ [0.6931, 0.9146][0.7597, 0.8985] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.7943, 0.9124][0.8326, 0.9029] \\ [0.6178, 0.8670][0.6170, 0.7597] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.8866, 0.9650][0.8326, 0.8706] \\ [0.6178, 0.7578][0.6968, 0.8123] \end{Bmatrix}^{0.4} \end{pmatrix} \right)$$

$$= ([0.3401, 0.5912], [0.2813, 0.5574])$$

$$\Theta_2 = \left(\sqrt{1 - \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(1 - \left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}}, \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(\left[\delta_{ij}^{l}, \delta_{ij}^{u} \right] \right)^{\omega_i} \right)^{\nu_j} \right)$$

$$= \left(\sqrt{1 - \begin{pmatrix} \begin{Bmatrix} [0.64, 0.91]^{0.1} [0.51, 0.75]^{0.2} \\ [0.51, 0.96]^{0.4} [0.51, 0.64]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.75, 0.91]^{0.1} [0.75, 0.84]^{0.2} \\ [0.75, 0.99]^{0.4} [0.75, 0.84]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.64, 0.96]^{0.1} [0.96, 0.99]^{0.2} \\ [0.51, 0.84]^{0.4} [0.36, 0.75]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.91, 0.96]^{0.1} [0.36, 0.96]^{0.2} \\ [0.51, 0.91]^{0.4} [0.51, 0.99]^{0.3} \end{Bmatrix}^{0.4} \end{pmatrix}}, \begin{pmatrix} \begin{Bmatrix} [0.5, 0.6]^{0.1} [0.2, 0.7]^{0.2} \\ [0.4, 0.5]^{0.4} [0.5, 0.8]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.5, 0.8]^{0.1} [0.1, 0.4]^{0.2} \\ [0.3, 0.7]^{0.4} [0.3, 0.6]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.1, 0.4]^{0.1} [0.1, 0.2]^{0.2} \\ [0.3, 0.8]^{0.4} [0.2, 0.6]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.3, 0.8]^{0.1} [0.3, 0.5]^{0.2} \\ [0.2, 0.6]^{0.4} [0.3, 0.6]^{0.3} \end{Bmatrix}^{0.4} \end{pmatrix} \right)$$

$$= \left(\sqrt{1 - \begin{pmatrix} \begin{Bmatrix} [0.9564, 0.9906][0.8740, 0.9441] \\ [0.7639, 0.9838][0.8171, 0.8747] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.9716, 0.9906][0.9441, 0.9657] \\ [0.8913, 0.9960][0.9173, 0.9490] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.9564, 0.9960][0.9919, 0.9980] \\ [0.7639, 0.9326][0.7360, 0.9173] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.9906, 0.9959][0.8152, 0.9919] \\ [0.7639, 0.9630][0.8171, 0.9970] \end{Bmatrix}^{0.4} \end{pmatrix}}, \begin{pmatrix} \begin{Bmatrix} [0.9330, 0.9502][0.7248, 0.9311] \\ [0.6931, 0.7579][0.8123, 0.9352] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.9330, 0.9779][0.6310, 0.8326] \\ [0.6178, 0.8670][0.6968, 0.8579] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.7943, 0.9124][0.6310, 0.7248] \\ [0.6178, 0.9146][0.6170, 0.8579] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.8866, 0.9780][0.7860, 0.8706] \\ [0.5253, 0.8152][0.6968, 0.8579] \end{Bmatrix}^{0.4} \end{pmatrix} \right)$$

$$= ([0.3470, 0.6811], [0.2713, 0.5864])$$

$$\Theta_3 = \left(\sqrt{1 - \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(1 - \left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}}, \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(\left[\delta_{ij}^{l}, \delta_{ij}^{u} \right] \right)^{\omega_i} \right)^{\nu_j} \right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\begin{array}{cc} \left\{ \begin{matrix} [0.84, 0.91]^{0.1} \, [0.64, 0.84]^{0.2} \\ [0.64, 0.75]^{0.4} \, [0.84, 0.91]^{0.3} \end{matrix} \right\}^{0.3} & \left\{ \begin{matrix} [0.64, 0.84]^{0.1} \, [0.91, 0.96]^{0.2} \\ [0.75, 0.91]^{0.4} \, [0.64, 0.96]^{0.3} \end{matrix} \right\}^{0.1} \\ \left\{ \begin{matrix} [0.84, 0.96]^{0.1} \, [0.51, 0.91]^{0.2} \\ [0.51, 0.91]^{0.4} \, [0.91, 0.99]^{0.3} \end{matrix} \right\}^{0.2} & \left\{ \begin{matrix} [0.51, 0.91]^{0.1} \, [0.84, 0.96]^{0.2} \\ [0.75, 0.96]^{0.4} \, [0.84, 0.91]^{0.3} \end{matrix} \right\}^{0.4} \end{array} \right)}, \\ \left(\begin{array}{cc} \left\{ \begin{matrix} [0.2, 0.7]^{0.1} \, [0.3, 0.4]^{0.2} \\ [0.4, 0.5]^{0.4} \, [0.3, 0.6]^{0.3} \end{matrix} \right\}^{0.3} & \left\{ \begin{matrix} [0.3, 0.7]^{0.1} \, [0.3, 0.5]^{0.2} \\ [0.5, 0.8]^{0.4} \, [0.2, 0.4]^{0.3} \end{matrix} \right\}^{0.1} \\ \left\{ \begin{matrix} [0.3, 0.4]^{0.1} \, [0.3, 0.5]^{0.2} \\ [0.3, 0.8]^{0.4} \, [0.5, 0.6]^{0.3} \end{matrix} \right\}^{0.2} & \left\{ \begin{matrix} [0.3, 0.7]^{0.1} \, [0.3, 0.5]^{0.2} \\ [0.3, 0.6]^{0.4} \, [0.3, 0.7]^{0.3} \end{matrix} \right\}^{0.4} \end{array} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\begin{array}{cc} \left\{ \begin{matrix} [0.9827, 0.9906][0.9146, 0.9657] \\ [0.8365, 0.8913][0.9490, 0.9721] \end{matrix} \right\}^{0.3} & \left\{ \begin{matrix} [0.9564, 0.9827][0.9813, 0.9919] \\ [0.8913, 0.9630][0.8747, 0.9878] \end{matrix} \right\}^{0.1} \\ \left\{ \begin{matrix} [0.9827, 0.9959][0.8740, 0.9813] \\ [0.7639, 0.9630][0.9721, 0.9970] \end{matrix} \right\}^{0.2} & \left\{ \begin{matrix} [0.9349, 0.9906][0.9657, 0.9919] \\ [0.8913, 0.9838][0.9490, 0.9721] \end{matrix} \right\}^{0.4} \end{array} \right)}, \\ \left(\begin{array}{cc} \left\{ \begin{matrix} [0.8513, 0.9649][0.7860, 0.8326] \\ [0.6931, 0.7579][0.6968, 0.8579] \end{matrix} \right\}^{0.3} & \left\{ \begin{matrix} [0.8865, 0.9650][0.7860, 0.8706] \\ [0.7579, 0.9146][0.6170, 0.7597] \end{matrix} \right\}^{0.1} \\ \left\{ \begin{matrix} [0.8866, 0.9124][0.7860, 0.8326] \\ [0.6178, 0.9146][0.8123, 0.8579] \end{matrix} \right\}^{0.2} & \left\{ \begin{matrix} [0.8866, 0.9650][0.7860, 0.8706] \\ [0.6178, 0.8152][0.6968, 0.8985] \end{matrix} \right\}^{0.4} \end{array} \right) \end{array} \right)$$

$$= ([0.3114, 0.5751], [0.3190, 0.5509])$$

$$\Theta_4 = \left(\sqrt{1 - \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(1 - \left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}}, \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(\left[\delta_{ij}^{l}, \delta_{ij}^{u} \right] \right)^{\omega_i} \right)^{\nu_j} \right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\begin{array}{cc} \left\{ \begin{matrix} [0.75, 0.91]^{0.1} \, [0.51, 0.84]^{0.2} \\ [0.75, 0.96]^{0.4} \, [0.51, 0.75]^{0.3} \end{matrix} \right\}^{0.3} & \left\{ \begin{matrix} [0.51, 0.96]^{0.1} \, [0.51, 0.84]^{0.2} \\ [0.51, 0.75]^{0.4} \, [0.75, 0.96]^{0.3} \end{matrix} \right\}^{0.1} \\ \left\{ \begin{matrix} [0.75, 0.96]^{0.1} \, [0.75, 0.99]^{0.2} \\ [0.84, 0.96]^{0.4} \, [0.75, 0.91]^{0.3} \end{matrix} \right\}^{0.2} & \left\{ \begin{matrix} [0.84, 0.96]^{0.1} \, [0.36, 0.75]^{0.2} \\ [0.51, 0.96]^{0.4} \, [0.75, 0.96]^{0.3} \end{matrix} \right\}^{0.4} \end{array} \right)}, \\ \left(\begin{array}{cc} \left\{ \begin{matrix} [0.2, 0.6]^{0.1} \, [0.2, 0.6]^{0.2} \\ [0.3, 0.6]^{0.4} \, [0.6, 0.8]^{0.3} \end{matrix} \right\}^{0.3} & \left\{ \begin{matrix} [0.3, 0.8]^{0.1} \, [0.1, 0.5]^{0.2} \\ [0.4, 0.5]^{0.4} \, [0.3, 0.4]^{0.3} \end{matrix} \right\}^{0.1} \\ \left\{ \begin{matrix} [0.1, 0.6]^{0.1} \, [0.2, 0.5]^{0.2} \\ [0.2, 0.7]^{0.4} \, [0.1, 0.5]^{0.3} \end{matrix} \right\}^{0.2} & \left\{ \begin{matrix} [0.5, 0.8]^{0.1} \, [0.2, 0.5]^{0.2} \\ [0.3, 0.6]^{0.4} \, [0.4, 0.5]^{0.3} \end{matrix} \right\}^{0.4} \end{array} \right) \end{array} \right)$$

$$= \left(\sqrt{1 - \begin{pmatrix} \begin{Bmatrix} [0.9716, 0.9906][0.8740, 0.9657] \\ [0.8913, 0.9838][0.8171, 0.9173] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.9349, 0.9959][0.8740, 0.9657] \\ [0.7639, 0.8913][0.9173, 0.9878] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.9716, 0.9959][0.9441, 0.9980] \\ [0.9326, 0.9838][0.9173, 0.9721] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.9827, 0.9959][0.8151, 0.9441] \\ [0.7639, 0.9838][0.9173, 0.9878] \end{Bmatrix}^{0.4} \end{pmatrix}}, \begin{pmatrix} \begin{Bmatrix} [0.8513, 0.9502][0.7248, 0.9029] \\ [0.6178, 0.8151][0.8579, 0.9352] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.8866, 0.9779][0.6310, 0.8706] \\ [0.6931, 0.7579][0.6968, 0.7597] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.7943, 0.9502][0.7248, 0.8706] \\ [0.5253, 0.8670][0.5012, 0.8122] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.9330, 0.9779][0.7248, 0.8706] \\ [0.6078, 0.8152][0.7597, 0.8123] \end{Bmatrix}^{0.4} \end{pmatrix} \right)$$

$$= ([0.3162, 0.7856], [0.2701, 0.6137]).$$

Step-4: Use the score function $S = \frac{(\kappa^l)^2 + (\kappa^u)^2 - (\delta^l)^2 - (\delta^u)^2}{2}$ for the interval-valued Pythagorean fuzzy soft set to calculate the score values for all alternatives. $S(\Theta_1) = 0.0377$, $S(\Theta_2) = 0.0834$, $S(\Theta_3) = 0.0113$, and $S(\Theta_4) = 0.0141$.

Step-5: From the above calculation, we get $S(\Theta_2) > S(\Theta_1) > S(\Theta_4) > S(\Theta_3)$, which shows that $\mathfrak{J}^2$ is the best alternative. So, $\mathfrak{J}^2 > \mathfrak{J}^1 > \mathfrak{J}^4 > \mathfrak{J}^3$.

4.2.2 By IVPFSWG Operator

Step-1: Obtain PFS decision matrices (Tables 1–4).

Step-2: Use the normalization formula to normalize the obtained PFS decision matrices (Tables 5–8).

Step-3: Apply the proposed IVPFSWG operator on the acquired data, we will obtain an opinion of the decision-makers

$$\Theta_1 = \left(\prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(\left[\kappa_{ij}^l, \kappa_{ij}^u \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(1 - \left[\delta_{ij}^l, \delta_{ij}^u \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)$$

$$= \left(\begin{pmatrix} \begin{Bmatrix} [0.2, 0.5]^{0.1} [0.7, 0.8]^{0.2} \\ [0.2, 0.5]^{0.4} [0.2, 0.6]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.2, 0.6]^{0.1} [0.1, 0.6]^{0.2} \\ [0.4, 0.8]^{0.4} [0.4, 0.7]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.1, 0.4]^{0.1} [0.4, 0.6]^{0.2} \\ [0.3, 0.7]^{0.4} [0.2, 0.4]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.3, 0.7]^{0.1} [0.4, 0.5]^{0.2} \\ [0.3, 0.5]^{0.4} [0.3, 0.5]^{0.3} \end{Bmatrix}^{0.4} \end{pmatrix}, \sqrt{1 - \begin{pmatrix} \begin{Bmatrix} [0.75, 0.84]^{0.1} [0.64, 0.75]^{0.2} \\ [0.64, 0.84]^{0.4} [0.84, 0.96]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.51, 0.96]^{0.1} [0.75, 0.84]^{0.2} \\ [0.91, 0.96]^{0.4} [0.75, 0.96]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.75, 0.91]^{0.1} [0.51, 0.96]^{0.2} \\ [0.51, 0.84]^{0.4} [0.75, 0.91]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.64, 0.84]^{0.1} [0.51, 0.91]^{0.2} \\ [0.64, 0.91]^{0.4} [0.64, 0.91]^{0.3} \end{Bmatrix}^{0.4} \end{pmatrix}} \right)$$

$$= \left(\frac{\left(\begin{Bmatrix} [0.8513, 0.9330][0.9311, 0.9564] \\ [0.5253, 0.7579][0.6170, 0.8579] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.8513, 0.9502][0.6310, 0.9029] \\ [0.6931, 0.9146][0.7597, 0.8985] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.7943, 0.9124][0.8326, 0.9029] \\ [0.6178, 0.8670][0.6170, 0.7597] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.8866, 0.9650][0.8326, 0.8706] \\ [0.0.6178, 0.7578][0.6968, 0.8123] \end{Bmatrix}^{0.4} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.9716, 0.9827][0.9146, 0.9441] \\ [0.8365, 0.9326][0.9490, 0.9878] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.9349, 0.9949][0.9441, 0.9657] \\ [0.9629, 0.9838][0.9173, 0.9876] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.9716, 0.9906][0.8740, 0.9919] \\ [0.7639, 0.9326][0.9173, 0.9721] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.9564, 0.9827][0.8740, 0.9813] \\ [0.8365, 0.9630][0.8747, 0.9721] \end{Bmatrix}^{0.4} \right)}} \right)$$

$$= ([0.7975, 0.8569], [0.6395, 0.7586])$$

$$\Theta_2 = \left(\prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(1 - \left[\delta_{ij}^{l}, \delta_{ij}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)$$

$$= \left(\frac{\left(\begin{Bmatrix} [0.2, 0.5]^{0.1} [0.7, 0.8]^{0.2} \\ [0.2, 0.5]^{0.4} [0.2, 0.6]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.2, 0.6]^{0.1} [0.1, 0.6]^{0.2} \\ [0.4, 0.8]^{0.4} [0.4, 0.7]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.1, 0.4]^{0.1} [0.4, 0.6]^{0.2} \\ [0.3, 0.7]^{0.4} [0.2, 0.4]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.3, 0.7]^{0.1} [0.4, 0.5]^{0.2} \\ [0.3, 0.5]^{0.4} [0.3, 0.5]^{0.3} \end{Bmatrix}^{0.4} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.75, 0.84]^{0.1} [0.64, 0.75]^{0.2} \\ [0.64, 0.84]^{0.4} [0.84, 0.96]^{0.3} \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.51, 0.96]^{0.1} [0.75, 0.84]^{0.2} \\ [0.91, 0.96]^{0.4} [0.75, 0.96]^{0.3} \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.75, 0.91]^{0.1} [0.51, 0.96]^{0.2} \\ [0.51, 0.84]^{0.4} [0.75, 0.91]^{0.3} \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.64, 0.84]^{0.1} [0.51, 0.91]^{0.2} \\ [0.64, 0.91]^{0.4} [0.64, 0.91]^{0.3} \end{Bmatrix}^{0.4} \right)}} \right)$$

$$= \left(\frac{\left(\begin{Bmatrix} [0.8513, 0.9330][0.9311, 0.9564] \\ [0.5253, 0.7579][0.6170, 0.8579] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.8513, 0.9502][0.6310, 0.9029] \\ [0.6931, 0.9146][0.7597, 0.8985] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.7943, 0.9124][0.8326, 0.9029] \\ [0.6178, 0.8670][0.6170, 0.7597] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.8866, 0.9650][0.8326, 0.8706] \\ [0.0.6178, 0.7578][0.6968, 0.8123] \end{Bmatrix}^{0.4} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.9716, 0.9827][0.9146, 0.9441] \\ [0.8365, 0.9326][0.9490, 0.9878] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.9349, 0.9949][0.9441, 0.9657] \\ [0.9629, 0.9838][0.9173, 0.9876] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.9716, 0.9906][0.8740, 0.9919] \\ [0.7639, 0.9326][0.9173, 0.9721] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.9564, 0.9827][0.8740, 0.9813] \\ [0.8365, 0.9630][0.8747, 0.9721] \end{Bmatrix}^{0.4} \right)}} \right)$$

$$= ([0.5643, 0.8978], [0.5206, 0.7452])$$

$$\Theta_3 = \left(\prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(1 - \left[\delta_{ij}^{l}, \delta_{ij}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)$$

$$
= \left(\begin{array}{c}
\left(\begin{array}{c}
\left\{ \begin{matrix} [0.2, 0.5]^{0.1} \, [0.7, 0.8]^{0.2} \\ [0.2, 0.5]^{0.4} \, [0.2, 0.6]^{0.3} \end{matrix} \right\}^{0.3}, \left\{ \begin{matrix} [0.2, 0.6]^{0.1} \, [0.1, 0.6]^{0.2} \\ [0.4, 0.8]^{0.4} \, [0.4, 0.7]^{0.3} \end{matrix} \right\}^{0.1} \\
\left\{ \begin{matrix} [0.1, 0.4]^{0.1} \, [0.4, 0.6]^{0.2} \\ [0.3, 0.7]^{0.4} \, [0.2, 0.4]^{0.3} \end{matrix} \right\}^{0.2} \left\{ \begin{matrix} [0.3, 0.7]^{0.1} \, [0.4, 0.5]^{0.2} \\ [0.3, 0.5]^{0.4} \, [0.3, 0.5]^{0.3} \end{matrix} \right\}^{0.4}
\end{array} \right), \\
\sqrt{1 - \left(\begin{array}{c}
\left\{ \begin{matrix} [0.75, 0.84]^{0.1} \, [0.64, 0.75]^{0.2} \\ [0.64, 0.84]^{0.4} \, [0.84, 0.96]^{0.3} \end{matrix} \right\}^{0.3} \left\{ \begin{matrix} [0.51, 0.96]^{0.1} \, [0.75, 0.84]^{0.2} \\ [0.91, 0.96]^{0.4} \, [0.75, 0.96]^{0.3} \end{matrix} \right\}^{0.1} \\
\left\{ \begin{matrix} [0.75, 0.91]^{0.1} \, [0.51, 0.96]^{0.2} \\ [0.51, 0.84]^{0.4} \, [0.75, 0.91]^{0.3} \end{matrix} \right\}^{0.2} \left\{ \begin{matrix} [0.64, 0.84]^{0.1} \, [0.51, 0.91]^{0.2} \\ [0.64, 0.91]^{0.4} \, [0.64, 0.91]^{0.3} \end{matrix} \right\}^{0.4}
\end{array} \right)}
\end{array} \right)
$$

$$
= \left(\begin{array}{c}
\left(\begin{array}{c}
\left\{ \begin{matrix} [0.8513, 0.9330][0.9311, 0.9564] \\ [0.5253, 0.7579][0.6170, 0.8579] \end{matrix} \right\}^{0.3} \left\{ \begin{matrix} [0.8513, 0.9502][0.6310, 0.9029] \\ [0.6931, 0.9146][0.7597, 0.8985] \end{matrix} \right\}^{0.1} \\
\left\{ \begin{matrix} [0.7943, 0.9124][0.8326, 0.9029] \\ [0.6178, 0.8670][0.6170, 0.7597] \end{matrix} \right\}^{0.2} \left\{ \begin{matrix} [0.8866, 0.9650][0.8326, 0.8706] \\ [0.0.6178, 0.7578][0.6968, 0.8123] \end{matrix} \right\}^{0.4}
\end{array} \right), \\
\sqrt{1 - \left(\begin{array}{c}
\left\{ \begin{matrix} [0.9716, 0.9827][0.9146, 0.9441] \\ [0.8365, 0.9326][0.9490, 0.9878] \end{matrix} \right\}^{0.3} \left\{ \begin{matrix} [0.9349, 0.9949][0.9441, 0.9657] \\ [0.9629, 0.9838][0.9173, 0.9876] \end{matrix} \right\}^{0.1} \\
\left\{ \begin{matrix} [0.9716, 0.9906][0.8740, 0.9919] \\ [0.7639, 0.9326][0.9173, 0.9721] \end{matrix} \right\}^{0.2} \left\{ \begin{matrix} [0.9564, 0.9827][0.8740, 0.9813] \\ [0.8365, 0.9630][0.8747, 0.9721] \end{matrix} \right\}^{0.4}
\end{array} \right)}
\end{array} \right)
$$

$$
= ([0.6325, 0.9658], [0.2365, 0.5263])
$$

$$
\Theta_4 = \left(\prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(\left[\kappa_{ij}^{l}, \kappa_{ij}^{u} \right] \right)^{\omega_i} \right)^{\nu_j}, \sqrt{1 - \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(1 - \left[\delta_{ij}^{l}, \delta_{ij}^{u} \right]^2 \right)^{\omega_i} \right)^{\nu_j}} \right)
$$

$$
= \left(\begin{array}{c}
\left(\begin{array}{c}
\left\{ \begin{matrix} [0.2, 0.5]^{0.1} \, [0.7, 0.8]^{0.2} \\ [0.2, 0.5]^{0.4} \, [0.2, 0.6]^{0.3} \end{matrix} \right\}^{0.3}, \left\{ \begin{matrix} [0.2, 0.6]^{0.1} \, [0.1, 0.6]^{0.2} \\ [0.4, 0.8]^{0.4} \, [0.4, 0.7]^{0.3} \end{matrix} \right\}^{0.1} \\
\left\{ \begin{matrix} [0.1, 0.4]^{0.1} \, [0.4, 0.6]^{0.2} \\ [0.3, 0.7]^{0.4} \, [0.2, 0.4]^{0.3} \end{matrix} \right\}^{0.2} \left\{ \begin{matrix} [0.3, 0.7]^{0.1} \, [0.4, 0.5]^{0.2} \\ [0.3, 0.5]^{0.4} \, [0.3, 0.5]^{0.3} \end{matrix} \right\}^{0.4}
\end{array} \right), \\
\sqrt{1 - \left(\begin{array}{c}
\left\{ \begin{matrix} [0.75, 0.84]^{0.1} \, [0.64, 0.75]^{0.2} \\ [0.64, 0.84]^{0.4} \, [0.84, 0.96]^{0.3} \end{matrix} \right\}^{0.3} \left\{ \begin{matrix} [0.51, 0.96]^{0.1} \, [0.75, 0.84]^{0.2} \\ [0.91, 0.96]^{0.4} \, [0.75, 0.96]^{0.3} \end{matrix} \right\}^{0.1} \\
\left\{ \begin{matrix} [0.75, 0.91]^{0.1} \, [0.51, 0.96]^{0.2} \\ [0.51, 0.84]^{0.4} \, [0.75, 0.91]^{0.3} \end{matrix} \right\}^{0.2} \left\{ \begin{matrix} [0.64, 0.84]^{0.1} \, [0.51, 0.91]^{0.2} \\ [0.64, 0.91]^{0.4} \, [0.64, 0.91]^{0.3} \end{matrix} \right\}^{0.4}
\end{array} \right)}
\end{array} \right)
$$

$$= \left(\frac{\left(\begin{Bmatrix} [0.8513, 0.9330][0.9311, 0.9564] \\ [0.5253, 0.7579][0.6170, 0.8579] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.8513, 0.9502][0.6310, 0.9029] \\ [0.6931, 0.9146][0.7597, 0.8985] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.7943, 0.9124][0.8326, 0.9029] \\ [0.6178, 0.8670][0.6170, 0.7597] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.8866, 0.9650][0.8326, 0.8706] \\ [0.0.6178, 0.7578][0.6968, 0.8123] \end{Bmatrix}^{0.4} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.9716, 0.9827][0.9146, 0.9441] \\ [0.8365, 0.9326][0.9490, 0.9878] \end{Bmatrix}^{0.3} \begin{Bmatrix} [0.9349, 0.9949][0.9441, 0.9657] \\ [0.9629, 0.9838][0.9173, 0.9876] \end{Bmatrix}^{0.1} \\ \begin{Bmatrix} [0.9716, 0.9906][0.8740, 0.9919] \\ [0.7639, 0.9326][0.9173, 0.9721] \end{Bmatrix}^{0.2} \begin{Bmatrix} [0.9564, 0.9827][0.8740, 0.9813] \\ [0.8365, 0.9630][0.8747, 0.9721] \end{Bmatrix}^{0.4} \right)}} \right)$$

$$= ([0.4525, 0.5469], [0.1253, 0.5263]).$$

Step-4: Use the score function $S = \frac{\left(\kappa^l\right)^2 + \left(\kappa^u\right)^2 - \left(\delta^l\right)^2 - \left(\delta^u\right)^2}{2}$ interval-valued for the Pythagorean fuzzy soft set to calculate the score values for all alternatives such as $S(\Theta_1) = 0.0524$, $S(\Theta_2) = 0.0754$, $S(\Theta_3) = 0.0241$, and $S(\Theta_4) = 0.0114$.

Step-5: From the above calculation, we get the ranking of alternatives $S(\Theta_2) > S(\Theta_1) > S(\Theta_3) > S(\Theta_4)$, which shows that $\mathfrak{I}^2$ is the best alternative. So, $\mathfrak{I}^2 > \mathfrak{I}^1 > \mathfrak{I}^3 > \mathfrak{I}^4$.

5 Comparative Studies

To highlight the effectiveness of the presented method, a comparison between the proposed model and prevailing methods is proposed in the following section.

5.1 Comparative Analysis with Interval-Valued Pythagorean Fuzzy Weighted Average Operator [28]

Step-1: Obtain an IVPF decision matrices (Tables 1–4).

Step-2: Use normalization formula to normalize the obtained IVPF decision matrices (Tables 5–8).

Step-3: Apply the IVPFWA operator on the acquired data, then we get the opinion of decision-makers.

As we have

$$IVPFWA(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \ldots\ldots \mathcal{M}_n) = \left(\sqrt{1 - \prod_{i=1}^{4} \left(1 - \left[\kappa_i^l, \kappa_i^u\right]^2 \right)^{\omega_i}}, \prod_{i=1}^{4} \left[\delta_i^l, \delta_i^u\right]^{\omega_i} \right)$$

$$\Theta_1 = \left(\begin{array}{c} \sqrt{1 - \left(\begin{Bmatrix} [0.75, 0.84]^{0.1} [0.64, 0.75]^{0.2} \\ [0.64, 0.84]^{0.4} [0.84, 0.96]^{0.3} \end{Bmatrix} \begin{Bmatrix} [0.51, 0.96]^{0.1} [0.75, 0.84]^{0.2} \\ [0.91, 0.96]^{0.4} [0.75, 0.96]^{0.3} \end{Bmatrix} \\ \begin{Bmatrix} [0.75, 0.91]^{0.1} [0.51, 0.96]^{0.2} \\ [0.51, 0.84]^{0.4} [0.75, 0.91]^{0.3} \end{Bmatrix} \begin{Bmatrix} [0.64, 0.84]^{0.1} [0.51, 0.91]^{0.2} \\ [0.64, 0.91]^{0.4} [0.64, 0.91]^{0.3} \end{Bmatrix} \right)}, \\ \left(\begin{Bmatrix} [0.2, 0.5]^{0.1} [0.7, 0.8]^{0.2} \\ [0.2, 0.5]^{0.4} [0.2, 0.6]^{0.3} \end{Bmatrix} \begin{Bmatrix} [0.2, 0.6]^{0.1} [0.1, 0.6]^{0.2} \\ [0.4, 0.8]^{0.4} [0.4, 0.7]^{0.3} \end{Bmatrix} \\ \begin{Bmatrix} [0.1, 0.4]^{0.1} [0.4, 0.6]^{0.2} \\ [0.3, 0.7]^{0.4} [0.2, 0.4]^{0.3} \end{Bmatrix} \begin{Bmatrix} [0.3, 0.7]^{0.1} [0.4, 0.5]^{0.2} \\ [0.3, 0.5]^{0.4} [0.3, 0.5]^{0.3} \end{Bmatrix} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\begin{array}{c} \left\{ \begin{array}{c} {[0.9716, 0.9827][0.9146, 0.9441]} \\ {[0.8365, 0.9326][0.9490, 0.9878]} \end{array} \right\} \left\{ \begin{array}{c} {[0.9349, 0.9949][0.9441, 0.9657]} \\ {[0.9629, 0.9838][0.9173, 0.9876]} \end{array} \right\} \\ \left\{ \begin{array}{c} {[0.9716, 0.9906][0.8740, 0.9919]} \\ {[0.7639, 0.9326][0.9173, 0.9721]} \end{array} \right\} \left\{ \begin{array}{c} {[0.9564, 0.9827][0.8740, 0.9813]} \\ {[0.8365, 0.9630][0.8747, 0.9721]} \end{array} \right\} \end{array} \right)}, \\ \left(\begin{array}{c} \left\{ \begin{array}{c} {[0.8513, 0.9330][0.9311, 0.9564]} \\ {[0.5253, 0.7579][0.6170, 0.8579]} \end{array} \right\} \left\{ \begin{array}{c} {[0.8513, 0.9502][0.6310, 0.9029]} \\ {[0.6931, 0.9146][0.7597, 0.8985]} \end{array} \right\} \\ \left\{ \begin{array}{c} {[0.7943, 0.9124][0.8326, 0.9029]} \\ {[0.6178, 0.8670][0.6170, 0.7597]} \end{array} \right\} \left\{ \begin{array}{c} {[0.8866, 0.9650][0.8326, 0.8706]} \\ {[0.0.6178, 0.7578][0.6968, 0.8123]} \end{array} \right\} \end{array} \right) \end{array} \right)$$

$$= ([0.6071, 0.8944], [0.0083, 0.1265])$$

$$\Theta_2 = \left(\begin{array}{c} \sqrt{1 - \left(\begin{array}{c} \left\{ \begin{array}{c} {[0.64, 0.91]^{0.1}\,[0.51, 0.75]^{0.2}} \\ {[0.51, 0.96]^{0.4}\,[0.51, 0.64]^{0.3}} \end{array} \right\} \left\{ \begin{array}{c} {[0.75, 0.91]^{0.1}\,[0.75, 0.84]^{0.2}} \\ {[0.75, 0.99]^{0.4}\,[0.75, 0.84]^{0.3}} \end{array} \right\} \\ \left\{ \begin{array}{c} {[0.64, 0.96]^{0.1}\,[0.19, 0.96]^{0.2}} \\ {[0.51, 0.96]^{0.4}\,[0.36, 0.75]^{0.3}} \end{array} \right\} \left\{ \begin{array}{c} {[0.91, 0.96]^{0.1}\,[0.19, 0.96]^{0.2}} \\ {[0.51, 0.84]^{0.4}\,[0.51, 0.99]^{0.3}} \end{array} \right\} \end{array} \right)}, \\ \left(\begin{array}{c} \left\{ \begin{array}{c} {[0.5, 0.6]^{0.1}\,[0.2, 0.7]^{0.2}} \\ {[0.4, 0.5]^{0.4}\,[0.5, 0.8]^{0.3}} \end{array} \right\} \left\{ \begin{array}{c} {[0.5, 0.8]^{0.1}\,[0.1, 0.4]^{0.2}} \\ {[0.3, 0.7]^{0.4}\,[0.3, 0.6]^{0.3}} \end{array} \right\} \\ \left\{ \begin{array}{c} {[0.1, 0.4]^{0.1}\,[0.1, 0.2]^{0.2}} \\ {[0.3, 0.8]^{0.4}\,[0.2, 0.6]^{0.3}} \end{array} \right\} \left\{ \begin{array}{c} {[0.3, 0.8]^{0.1}\,[0.3, 0.5]^{0.2}} \\ {[0.2, 0.6]^{0.4}\,[0.3, 0.6]^{0.3}} \end{array} \right\} \end{array} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\begin{array}{c} \left\{ \begin{array}{c} {[0.9564, 0.9906][0.8740, 0.9441]} \\ {[0.7639, 0.9838][0.8171, 0.8747]} \end{array} \right\} \left\{ \begin{array}{c} {[0.9716, 0.9906][0.9441, 0.9657]} \\ {[0.8913, 0.9960][0.9173, 0.9490]} \end{array} \right\} \\ \left\{ \begin{array}{c} {[0.9564, 0.9960][0.7174, 0.9918]} \\ {[0.7639, 0.9838][0.7360, 0.9173]} \end{array} \right\} \left\{ \begin{array}{c} {[0.9906, 0.9959][0.7174, 0.9919]} \\ {[0.7639, 0.9326][0.8171, 0.9970]} \end{array} \right\} \end{array} \right)}, \\ \left(\begin{array}{c} \left\{ \begin{array}{c} {[0.9930, 0.9502][0.7248, 0.9311]} \\ {[0.6931, 0.7579][0.8123, 0.9352]} \end{array} \right\} \left\{ \begin{array}{c} {[0.9330, 0.9780][0.6309, 0.8326]} \\ {[0.6178, 0.8670][0.6968, 0.8579]} \end{array} \right\} \\ \left\{ \begin{array}{c} {[0.7943, 0.9124][0.6310, 0.7248]} \\ {[0.6178, 0.9146][0.6170, 0.8579]} \end{array} \right\} \left\{ \begin{array}{c} {[0.8866, 0.9779][0.7860, 0.8706]} \\ {[0.5253, 0.8152][0.6968, 0.8579]} \end{array} \right\} \end{array} \right) \end{array} \right)$$

$$= ([0.6357, 0.9659], [0.0050, 0.1174])$$

$$\Theta_3 = \left(\begin{array}{c} \sqrt{1 - \left(\begin{array}{c} \left\{ \begin{array}{c} {[0.84, 0.91]^{0.1}\,[0.64, 0.84]^{0.2}} \\ {[0.64, 0.75]^{0.4}\,[0.84, 0.91]^{0.3}} \end{array} \right\} \left\{ \begin{array}{c} {[0.64, 0.84]^{0.1}\,[0.91, 0.96]^{0.2}} \\ {[0.75, 0.91]^{0.4}\,[0.91, 0.99]^{0.3}} \end{array} \right\} \\ \left\{ \begin{array}{c} {[0.84, 0.96]^{0.1}\,[0.51, 0.91]^{0.2}} \\ {[0.51, 0.91]^{0.4}\,[0.91, 0.99]^{0.3}} \end{array} \right\} \left\{ \begin{array}{c} {[0.51, 0.91]^{0.1}\,[0.84, 0.96]^{0.2}} \\ {[0.75, 0.96]^{0.4}\,[0.84, 0.91]^{0.3}} \end{array} \right\} \end{array} \right)}, \\ \left(\begin{array}{c} \left\{ \begin{array}{c} {[0.2, 0.7]^{0.1}\,[0.3, 0.4]^{0.2}} \\ {[0.4, 0.5]^{0.4}\,[0.3, 0.6]^{0.3}} \end{array} \right\} \left\{ \begin{array}{c} {[0.3, 0.7]^{0.1}\,[0.3, 0.5]^{0.2}} \\ {[0.5, 0.8]^{0.4}\,[0.2, 0.4]^{0.3}} \end{array} \right\} \\ \left\{ \begin{array}{c} {[0.3, 0.4]^{0.1}\,[0.3, 0.5]^{0.2}} \\ {[0.3, 0.8]^{0.4}\,[0.5, 0.6]^{0.3}} \end{array} \right\} \left\{ \begin{array}{c} {[0.3, 0.7]^{0.1}\,[0.3, 0.5]^{0.2}} \\ {[0.3, 0.6]^{0.4}\,[0.3, 0.7]^{0.3}} \end{array} \right\} \end{array} \right) \end{array} \right)$$

$$= \left(\begin{matrix} \sqrt{1 - \left(\begin{matrix} \left\{ \begin{matrix} [0.9827, 0.9906][0.9146, 0.9657] \\ [0.8365, 0.8913][0.9490, 0.9721] \end{matrix} \right\} & \left\{ \begin{matrix} [0.9564, 0.9827][0.9813, 0.9919] \\ [0.8913, 0.9630][0.9721, 0.9970] \end{matrix} \right\} \\ \left\{ \begin{matrix} [0.9827, 0.9959][0.8740, 0.9813] \\ [0.7639, 0.9630][0.9721, 0.9970] \end{matrix} \right\} & \left\{ \begin{matrix} [0.9349, 0.9906][0.9657, 0.9919] \\ [0.8913, 0.9838][0.9490, 0.9721] \end{matrix} \right\} \end{matrix} \right)}, \\ \left(\begin{matrix} \left\{ \begin{matrix} [0.8513, 0.9650][0.7860, 0.8326] \\ [0.6931, 0.7579][0.6968, 0.8579] \end{matrix} \right\} & \left\{ \begin{matrix} [0.8866, 0.9650][0.7860, 0.8706] \\ [0.5759, 0.9146][0.6170, 0.7597] \end{matrix} \right\} \\ \left\{ \begin{matrix} [0.8866, 0.9124][0.7860, 0.8706] \\ [0.6178, 0.9146][0.8123, 0.8579] \end{matrix} \right\} & \left\{ \begin{matrix} [0.8866, 0.9650][0.7860, 0.8706] \\ [0.6178, 0.8152][0.6968, 0.8985] \end{matrix} \right\} \end{matrix} \right) \end{matrix} \right)$$

$$= ([0.3652, 0.6523], [0.5236, 0.6524])$$

$$\Theta_4 = \left(\begin{matrix} \sqrt{1 - \left(\begin{matrix} \left\{ \begin{matrix} [0.64, 0.91]^{0.1}\, [0.51, 0.75]^{0.2} \\ [0.51, 0.96]^{0.4}\, [0.51, 0.64]^{0.3} \end{matrix} \right\} & \left\{ \begin{matrix} [0.75, 0.91]^{0.1}\, [0.75, 0.84]^{0.2} \\ [0.75, 0.99]^{0.4}\, [0.75, 0.84]^{0.3} \end{matrix} \right\} \\ \left\{ \begin{matrix} [0.64, 0.96]^{0.1}\, [0.19, 0.96]^{0.2} \\ [0.51, 0.96]^{0.4}\, [0.36, 0.75]^{0.3} \end{matrix} \right\} & \left\{ \begin{matrix} [0.91, 0.96]^{0.1}\, [0.19, 0.96]^{0.2} \\ [0.51, 0.84]^{0.4}\, [0.51, 0.99]^{0.3} \end{matrix} \right\} \end{matrix} \right)}, \\ \left(\begin{matrix} \left\{ \begin{matrix} [0.5, 0.6]^{0.1}\, [0.2, 0.7]^{0.2} \\ [0.4, 0.5]^{0.4}\, [0.5, 0.8]^{0.3} \end{matrix} \right\} & \left\{ \begin{matrix} [0.5, 0.8]^{0.1}\, [0.1, 0.4]^{0.2} \\ [0.3, 0.7]^{0.4}\, [0.3, 0.6]^{0.3} \end{matrix} \right\} \\ \left\{ \begin{matrix} [0.1, 0.4]^{0.1}\, [0.1, 0.2]^{0.2} \\ [0.3, 0.8]^{0.4}\, [0.2, 0.6]^{0.3} \end{matrix} \right\} & \left\{ \begin{matrix} [0.3, 0.8]^{0.1}\, [0.3, 0.5]^{0.2} \\ [0.2, 0.6]^{0.4}\, [0.3, 0.6]^{0.3} \end{matrix} \right\} \end{matrix} \right) \end{matrix} \right)$$

$$= \left(\begin{matrix} \sqrt{1 - \left(\begin{matrix} \left\{ \begin{matrix} [0.9564, 0.9906][0.8740, 0.9441] \\ [0.7639, 0.9838][0.8171, 0.8747] \end{matrix} \right\} & \left\{ \begin{matrix} [0.9716, 0.9906][0.9441, 0.9657] \\ [0.8913, 0.9960][0.9173, 0.9490] \end{matrix} \right\} \\ \left\{ \begin{matrix} [0.9564, 0.9960][0.7174, 0.9918] \\ [0.7639, 0.9838][0.7360, 0.9173] \end{matrix} \right\} & \left\{ \begin{matrix} [0.9906, 0.9959][0.7174, 0.9919] \\ [0.7639, 0.9326][0.8171, 0.9970] \end{matrix} \right\} \end{matrix} \right)}, \\ \left(\begin{matrix} \left\{ \begin{matrix} [0.9930, 0.9502][0.7248, 0.9311] \\ [0.6931, 0.7579][0.8123, 0.9352] \end{matrix} \right\} & \left\{ \begin{matrix} [0.9330, 0.9780][0.6309, 0.8326] \\ [0.6178, 0.8670][0.6968, 0.8579] \end{matrix} \right\} \\ \left\{ \begin{matrix} [0.7943, 0.9124][0.6310, 0.7248] \\ [0.6178, 0.9146][0.6170, 0.8579] \end{matrix} \right\} & \left\{ \begin{matrix} [0.8866, 0.9779][0.7860, 0.8706] \\ [0.5253, 0.8152][0.6968, 0.8579] \end{matrix} \right\} \end{matrix} \right) \end{matrix} \right)$$

$$= ([0.6357, 0.9659], [0.0050, 0.1174])$$

Step-4: Use the score function $S = \frac{\left(\kappa^l\right)^2 + (\kappa^u)^2 - \left(\delta^l\right)^2 - (\delta^u)^2}{2}$ for IVPFS to calculate the score values for all alternatives.

$$S(\Theta_1) = \frac{(0.4575)^2 + (0.8569)^2 - (0.4595)^2 - (0.7586)^2}{2} = 0.0154$$

$$S(\Theta_2) = \frac{(0.6543)^2 + (0.8978)^2 - (0.5206)^2 - (0.7452)^2}{2} = 0.0251$$

$$S(\Theta_3) = \frac{(0.6565)^2 + (0.9548)^2 - (0.2365)^2 - (0.3663)^2}{2} = 0.0198$$

$$S(\Theta_4) = \frac{(0.4545)^2 + (0.5459)^2 - (0.8553)^2 - (0.2563)^2}{2} = 0.0247$$

Step-5: Ranking of alternatives $S(\Theta_2) > S(\Theta_4) > S(\Theta_3) > S(\Theta_1)$. So, $\mathfrak{I}^2 > \mathfrak{I}^4 > \mathfrak{I}^3 > \mathfrak{I}^1$. Hence, the best alternative is $\mathfrak{I}^2$.

5.2 Comparison with Interval-Valued Pythagorean Fuzzy Weighted Geometric Operator [28]

Step-1: Obtain an IVPF decision matrices (Tables 1–4).

Step-2: Use normalization formula to normalize the obtained IVPF decision matrices (Tables 5–8).

Step-3: Apply the IVPFWG operator on the acquired data, then we get the opinion of decision-makers.

As we have

$$IVPFWG\left(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \ldots\ldots \mathcal{M}_n\right) = \left(\prod_{i=1}^{4}\left[\kappa_i^l, \kappa_i^u\right]^{\omega_i}, \sqrt{1-\prod_{i=1}^{4}\left(1-\left[\delta_i^l, \delta_i^u\right]^2\right)^{\omega_i}}\right)$$

$$\Theta_1 = \left(\frac{\begin{pmatrix} \begin{Bmatrix} [0.2,0.5]^{0.1}\,[0.7,0.8]^{0.2} \\ [0.2,0.5]^{0.4}\,[0.2,0.6]^{0.3} \end{Bmatrix} & \begin{Bmatrix} [0.2,0.6]^{0.1}\,[0.1,0.6]^{0.2} \\ [0.4,0.8]^{0.4}\,[0.4,0.7]^{0.3} \end{Bmatrix} \\ \begin{Bmatrix} [0.1,0.4]^{0.1}\,[0.4,0.6]^{0.2} \\ [0.3,0.7]^{0.4}\,[0.2,0.4]^{0.3} \end{Bmatrix} & \begin{Bmatrix} [0.3,0.7]^{0.1}\,[0.4,0.5]^{0.2} \\ [0.3,0.5]^{0.4}\,[0.3,0.5]^{0.3} \end{Bmatrix} \end{pmatrix},}{\sqrt{1-\begin{pmatrix} \begin{Bmatrix} [0.75,0.84]^{0.1}\,[0.64,0.75]^{0.2} \\ [0.64,0.84]^{0.4}\,[0.84,0.96]^{0.3} \end{Bmatrix} & \begin{Bmatrix} [0.51,0.96]^{0.1}\,[0.75,0.84]^{0.2} \\ [0.91,0.96]^{0.4}\,[0.75,0.96]^{0.3} \end{Bmatrix} \\ \begin{Bmatrix} [0.75,0.91]^{0.1}\,[0.51,0.96]^{0.2} \\ [0.51,0.84]^{0.4}\,[0.75,0.91]^{0.3} \end{Bmatrix} & \begin{Bmatrix} [0.64,0.84]^{0.1}\,[0.51,0.91]^{0.2} \\ [0.64,0.91]^{0.4}\,[0.64,0.91]^{0.3} \end{Bmatrix} \end{pmatrix}}}\right)$$

$$= \left(\frac{\begin{pmatrix} \begin{Bmatrix} [0.8513,0.9330][0.9311,0.9564] \\ [0.5253,0.7579][0.6170,0.8579] \end{Bmatrix} & \begin{Bmatrix} [0.8513,0.9502][0.6310,0.9029] \\ [0.6931,0.9146][0.7597,0.8985] \end{Bmatrix} \\ \begin{Bmatrix} [0.7943,0.9124][0.8326,0.9029] \\ [0.6178,0.8670][0.6170,0.7597] \end{Bmatrix} & \begin{Bmatrix} [0.8866,0.9650][0.8326,0.8706] \\ [0.0.6178,0.7578][0.6968,0.8123] \end{Bmatrix} \end{pmatrix},}{\sqrt{1-\begin{pmatrix} \begin{Bmatrix} [0.9716,0.9827][0.9146,0.9441] \\ [0.8365,0.9326][0.9490,0.9878] \end{Bmatrix} & \begin{Bmatrix} [0.9349,0.9949][0.9441,0.9657] \\ [0.9629,0.9838][0.9173,0.9876] \end{Bmatrix} \\ \begin{Bmatrix} [0.9716,0.9906][0.8740,0.9919] \\ [0.7639,0.9326][0.9173,0.9721] \end{Bmatrix} & \begin{Bmatrix} [0.9564,0.9827][0.8740,0.9813] \\ [0.8365,0.9630][0.8747,0.9721] \end{Bmatrix} \end{pmatrix}}}\right)$$

$$= ([0.0083, 0.1265], [0.6071, 0.8944])$$

$$\Theta_2 = \left(\frac{\begin{pmatrix} \begin{Bmatrix} [0.5,0.6]^{0.1}\,[0.2,0.7]^{0.2} \\ [0.4,0.5]^{0.4}\,[0.5,0.8]^{0.3} \end{Bmatrix} & \begin{Bmatrix} [0.5,0.8]^{0.1}\,[0.1,0.4]^{0.2} \\ [0.3,0.7]^{0.4}\,[0.3,0.6]^{0.3} \end{Bmatrix} \\ \begin{Bmatrix} [0.1,0.4]^{0.1}\,[0.1,0.2]^{0.2} \\ [0.3,0.8]^{0.4}\,[0.2,0.6]^{0.3} \end{Bmatrix} & \begin{Bmatrix} [0.3,0.8]^{0.1}\,[0.3,0.5]^{0.2} \\ [0.2,0.6]^{0.4}\,[0.3,0.6]^{0.3} \end{Bmatrix} \end{pmatrix},}{\sqrt{1-\begin{pmatrix} \begin{Bmatrix} [0.64,0.91]^{0.1}\,[0.51,0.75]^{0.2} \\ [0.51,0.96]^{0.4}\,[0.51,0.64]^{0.3} \end{Bmatrix} & \begin{Bmatrix} [0.75,0.91]^{0.1}\,[0.75,0.84]^{0.2} \\ [0.75,0.99]^{0.4}\,[0.75,0.84]^{0.3} \end{Bmatrix} \\ \begin{Bmatrix} [0.64,0.96]^{0.1}\,[0.19,0.96]^{0.2} \\ [0.51,0.96]^{0.4}\,[0.36,0.75]^{0.3} \end{Bmatrix} & \begin{Bmatrix} [0.91,0.96]^{0.1}\,[0.19,0.96]^{0.2} \\ [0.51,0.84]^{0.4}\,[0.51,0.99]^{0.3} \end{Bmatrix} \end{pmatrix}}}\right)$$

$$= \left(\frac{\left(\begin{Bmatrix} [0.9930, 0.9502][0.7248, 0.9311] \\ [0.6931, 0.7579][0.8123, 0.9352] \end{Bmatrix} \begin{Bmatrix} [0.9330, 0.9780][0.6309, 0.8326] \\ [0.6178, 0.8670][0.6968, 0.8579] \end{Bmatrix} \\ \begin{Bmatrix} [0.7943, 0.9124][0.6310, 0.7248] \\ [0.6178, 0.9146][0.6170, 0.8579] \end{Bmatrix} \begin{Bmatrix} [0.8866, 0.9779][0.7860, 0.8706] \\ [0.5253, 0.8152][0.6968, 0.8579] \end{Bmatrix} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.9564, 0.9906][0.8740, 0.9441] \\ [0.7639, 0.9838][0.8171, 0.8747] \end{Bmatrix} \begin{Bmatrix} [0.9716, 0.9906][0.9441, 0.9657] \\ [0.8913, 0.9960][0.9173, 0.9490] \end{Bmatrix} \\ \begin{Bmatrix} [0.9564, 0.9960][0.7174, 0.9918] \\ [0.7639, 0.9838][0.7360, 0.9173] \end{Bmatrix} \begin{Bmatrix} [0.9906, 0.9959][0.7174, 0.9919] \\ [0.7639, 0.9326][0.8171, 0.9970] \end{Bmatrix} \right)}} \right)$$

$$= ([0.0050, 0.1174], [0.6357, 0.9659])$$

$$\Theta_3 = \left(\frac{\left(\begin{Bmatrix} [0.2, 0.7]^{0.1} [0.3, 0.4]^{0.2} \\ [0.4, 0.5]^{0.4} [0.3, 0.6]^{0.3} \end{Bmatrix} \begin{Bmatrix} [0.3, 0.7]^{0.1} [0.3, 0.5]^{0.2} \\ [0.5, 0.8]^{0.4} [0.2, 0.4]^{0.3} \end{Bmatrix} \\ \begin{Bmatrix} [0.3, 0.4]^{0.1} [0.3, 0.5]^{0.2} \\ [0.3, 0.8]^{0.4} [0.5, 0.6]^{0.3} \end{Bmatrix} \begin{Bmatrix} [0.3, 0.7]^{0.1} [0.3, 0.5]^{0.2} \\ [0.3, 0.6]^{0.4} [0.3, 0.7]^{0.3} \end{Bmatrix} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.84, 0.91]^{0.1} [0.64, 0.84]^{0.2} \\ [0.64, 0.75]^{0.4} [0.84, 0.91]^{0.3} \end{Bmatrix} \begin{Bmatrix} [0.64, 0.84]^{0.1} [0.91, 0.96]^{0.2} \\ [0.75, 0.91]^{0.4} [0.91, 0.99]^{0.3} \end{Bmatrix} \\ \begin{Bmatrix} [0.84, 0.96]^{0.1} [0.51, 0.91]^{0.2} \\ [0.51, 0.91]^{0.4} [0.91, 0.99]^{0.3} \end{Bmatrix} \begin{Bmatrix} [0.51, 0.91]^{0.1} [0.84, 0.96]^{0.2} \\ [0.75, 0.96]^{0.4} [0.84, 0.91]^{0.3} \end{Bmatrix} \right)}} \right)$$

$$= \left(\frac{\left(\begin{Bmatrix} [0.8513, 0.9650][0.7860, 0.8326] \\ [0.6931, 0.7579][0.6968, 0.8579] \end{Bmatrix} \begin{Bmatrix} [0.8866, 0.9650][0.7860, 0.8706] \\ [0.5759, 0.9146][0.6170, 0.7597] \end{Bmatrix} \\ \begin{Bmatrix} [0.8866, 0.9124][0.7860, 0.8706] \\ [0.6178, 0.9146][0.8123, 0.8579] \end{Bmatrix} \begin{Bmatrix} [0.8866, 0.9650][0.7860, 0.8706] \\ [0.6178, 0.8152][0.6968, 0.8985] \end{Bmatrix} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.9827, 0.9906][0.9146, 0.9657] \\ [0.8365, 0.8913][0.9490, 0.9721] \end{Bmatrix} \begin{Bmatrix} [0.9564, 0.9827][0.9813, 0.9919] \\ [0.8913, 0.9630][0.9721, 0.9970] \end{Bmatrix} \\ \begin{Bmatrix} [0.9827, 0.9959][0.8740, 0.9813] \\ [0.7639, 0.9630][0.9721, 0.9970] \end{Bmatrix} \begin{Bmatrix} [0.9349, 0.9906][0.9657, 0.9919] \\ [0.8913, 0.9838][0.9490, 0.9721] \end{Bmatrix} \right)}} \right)$$

$$= ([0.2565\ 0.5265], [0.2336, 0.4524])$$

$$\Theta_4 = \left(\frac{\left(\begin{Bmatrix} [0.5, 0.6]^{0.1} [0.2, 0.7]^{0.2} \\ [0.4, 0.5]^{0.4} [0.5, 0.8]^{0.3} \end{Bmatrix} \begin{Bmatrix} [0.5, 0.8]^{0.1} [0.1, 0.4]^{0.2} \\ [0.3, 0.7]^{0.4} [0.3, 0.6]^{0.3} \end{Bmatrix} \\ \begin{Bmatrix} [0.1, 0.4]^{0.1} [0.1, 0.2]^{0.2} \\ [0.3, 0.8]^{0.4} [0.2, 0.6]^{0.3} \end{Bmatrix} \begin{Bmatrix} [0.3, 0.8]^{0.1} [0.3, 0.5]^{0.2} \\ [0.2, 0.6]^{0.4} [0.3, 0.6]^{0.3} \end{Bmatrix} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.64, 0.91]^{0.1} [0.51, 0.75]^{0.2} \\ [0.51, 0.96]^{0.4} [0.51, 0.64]^{0.3} \end{Bmatrix} \begin{Bmatrix} [0.75, 0.91]^{0.1} [0.75, 0.84]^{0.2} \\ [0.75, 0.99]^{0.4} [0.75, 0.84]^{0.3} \end{Bmatrix} \\ \begin{Bmatrix} [0.64, 0.96]^{0.1} [0.19, 0.96]^{0.2} \\ [0.51, 0.96]^{0.4} [0.36, 0.75]^{0.3} \end{Bmatrix} \begin{Bmatrix} [0.91, 0.96]^{0.1} [0.19, 0.96]^{0.2} \\ [0.51, 0.84]^{0.4} [0.51, 0.99]^{0.3} \end{Bmatrix} \right)}} \right)$$

$$= \left(\frac{\left(\begin{Bmatrix} [0.9930, 0.9502][0.7248, 0.9311] \\ [0.6931, 0.7579][0.8123, 0.9352] \\ [0.7943, 0.9124][0.6310, 0.7248] \\ [0.6178, 0.9146][0.6170, 0.8579] \end{Bmatrix} \begin{Bmatrix} [0.9330, 0.9780][0.6309, 0.8326] \\ [0.6178, 0.8670][0.6968, 0.8579] \\ [0.8866, 0.9779][0.7860, 0.8706] \\ [0.5253, 0.8152][0.6968, 0.8579] \end{Bmatrix} \right),}{\sqrt{1 - \left(\begin{Bmatrix} [0.9564, 0.9906][0.8740, 0.9441] \\ [0.7639, 0.9838][0.8171, 0.8747] \\ [0.9564, 0.9960][0.7174, 0.9918] \\ [0.7639, 0.9838][0.7360, 0.9173] \end{Bmatrix} \begin{Bmatrix} [0.9716, 0.9906][0.9441, 0.9657] \\ [0.8913, 0.9960][0.9173, 0.9490] \\ [0.9906, 0.9959][0.7174, 0.9919] \\ [0.7639, 0.9326][0.8171, 0.9970] \end{Bmatrix} \right)}} \right)$$

$$= ([0.6357, 0.9659], [0.2564, 0.6585]).$$

Step-4: Use the score function $S = \frac{\left(\kappa^l\right)^2 + (\kappa^u)^2 - \left(\delta^l\right)^2 - (\delta^u)^2}{2}$ for IVPFS to calculate the score values for all alternatives.

$$S(\Theta_1) = \frac{(0.4575)^2 + (0.8569)^2 - (0.4595)^2 - (0.7586)^2}{2} = 0.030264$$

$$S(\Theta_2) = \frac{(0.6543)^2 + (0.8978)^2 - (0.5206)^2 - (0.7452)^2}{2} = 0.0856$$

$$S(\Theta_3) = \frac{(0.6565)^2 + (0.9548)^2 - (0.2365)^2 - (0.3663)^2}{2} = 0.0786$$

$$S(\Theta_4) = \frac{(0.6357)^2 + (0.9659)^2 - (0.2564)^2 - (0.6585)^2}{2} = 0.0475$$

Step-5: Ranking of alternatives, $S(\Theta_2) > S(\Theta_3) > S(\Theta_4) > S(\Theta_1)$. So, $\mathfrak{I}^2 > \mathfrak{I}^3 > \mathfrak{I}^1 > \mathfrak{I}^4$. Hence, the best alternative is $\mathfrak{I}^2$.

Similarly, we can get the outcomes utilizing several other existing operators for comparative studies.

5.3 Comparative Analysis

To verify the effectiveness of the proposed method, we compare the obtained results with some existing methods under the environment of IVPFS and IVIFSS. A summary of all results is given in Table 9. Zulqarnain et al. [17] developed aggregation operators for IVIFSS that are unable to accommodate the decision-makers choices when the sum of upper membership and nonmembership values of the parameters exceeds one. Peng et al. [27] interval-valued Pythagorean fuzzy weighted average operator and Rahman et al. [28] interval-valued Pythagorean fuzzy weighted geometric operator cannot handle the parametrized values of the alternatives. Furthermore, if only one parameter is supposed rather than more than one parameter, the interval-valued Pythagorean fuzzy soft set reduces to the interval-valued Pythagorean fuzzy set. Similarly, if the sum of upper values of membership and nonmembership degree is less or equal to 1. Then, IVPFSS reduced to IVIFSS. Thus, IVPFSS is the most generalized form of interval-valued Pythagorean fuzzy set. Hence, based on the above-mentioned facts, admittedly, the proposed operators in this paper are more powerful, reliable, and successful.

Table 9: Comparison of proposed operators with some existing operators

Approach	$\mathfrak{J}^1$	$\mathfrak{J}^2$	$\mathfrak{J}^3$	$\mathfrak{J}^4$	Alternatives ranking
IVPFWA [28]	0.0154	0.0251	0.0198	0.0247	$\mathfrak{J}^2 > \mathfrak{J}^4 > \mathfrak{J}^3 > \mathfrak{J}^1$
IVPFWG [28]	0.0364	0.0856	0.0786	0.0475	$\mathfrak{J}^2 > \mathfrak{J}^3 > \mathfrak{J}^1 > \mathfrak{J}^4$
IVIFSWA [18]	0.0235	0.0723	0.0584	0.2530	$\mathfrak{J}^2 > \mathfrak{J}^3 > \mathfrak{J}^1 > \mathfrak{J}^4$
IVIFSWG [18]	0.2365	0.7234	0.5840	0.6525	$\mathfrak{J}^2 > \mathfrak{J}^4 > \mathfrak{J}^3 > \mathfrak{J}^1$
Proposed IVPFSWA	0.0377	0.0834	0.0113	0.0141	$\mathfrak{J}^2 > \mathfrak{J}^1 > \mathfrak{J}^4 > \mathfrak{J}^3$
Proposed IVPFSWG	0.0524	0.0754	0.0241	0.0114	$\mathfrak{J}^2 > \mathfrak{J}^1 > \mathfrak{J}^3 > \mathfrak{J}^4$

6 Conclusion

In this work, we have introduced two novel aggregation operators such as IVPFSWA and IPFSWG operators. Firstly, we defined operational laws under an interval-valued Pythagorean fuzzy soft environment. Based on these operational laws, we developed the aggregation operators for IVPFSS such as IVPFSWA and IVPFSWG operators with their desirable properties. Furthermore, a DM approach has been established to resolve multi-attribute group decision-making (MAGDM) problems based on presented aggregation operators. To ensure the validity of the established technique, a comprehensive numerical example has been presented. To verify the effectiveness of the proposed method, a comparative analysis with some existing methods is presented. Finally, based on obtained results, it has been concluded that the proposed method in this research is the most feasible and successful method for the MAGDM problem.

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