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ARTICLE

Some Identities of the Degenerate Poly-Cauchy and Unipoly Cauchy Polynomials of the Second Kind

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ABSTRACT

In this paper, we introduce modified degenerate polyexponential Cauchy (or poly-Cauchy) polynomials and numbers of the second kind and investigate some identities of these polynomials. We derive recurrence relations and the relationship between special polynomials and numbers. Also, we introduce modified degenerate unipoly-Cauchy polynomials of the second kind and derive some fruitful properties of these polynomials. In addition, positive associated beautiful zeros and graphical representations are displayed with the help of Mathematica.

KEYWORDS

Modified degenerate polyexponential functions; modified degenerate polyexponential Cauchy (or poly-Cauchy) polynomials of the second kind; degenerate unipoly-Cauchy polynomials of the second kind

1 Introduction

Recently, many mathematicians, specifically Carlitz [1,2], Kim et al. [3–5], Kim et al. [6,7], Sharma et al. [8,9], Khan et al. [10–13], and Muhiuddin et al. [14–17] have studied and added diverse degenerate versions of many special polynomials and numbers (like as degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Daehee polynomials, degenerate Fubini polynomials, degenerate Stirling numbers of the first and second kind, and so on). In this paper, we focus on modified degenerate polyexponential Cauchy (or poly-Cauchy) polynomials and the numbers of the second type. The purpose of this paper is to introduce a degenerate model of the poly-Cauchy polynomials and numbers of the second type, the so-called degenerate poly-Cauchy polynomials, and numbers of the second type, constructed from the degenerate polyexponential feature. We derive some express expressions and identities for the one's numbers and polynomials.

Let $C_j(\xi)$ be the Cauchy polynomials which are given by the following generating function (see [18–22])

$$\int_0^1 (1+z)^{\xi+\eta} d\eta = \frac{z}{\log(1+z)}(1+z)^{\xi} = \sum_{j=0}^{\infty} C_j(\xi)\frac{z^j}{j!}. \tag{1}$$

In the case when $\xi = 0, C_j = C_j(0)$ are the Cauchy numbers.

The Bernoulli polynomials of order α are given by

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{\xi z} = \sum_{j=0}^{\infty} B_j^{(\alpha)}(\xi)\frac{z^j}{j!}, \text{(see [21,23])}. \tag{2}$$

For $\xi = 0$, $B_j^{(\alpha)} = B_j^{(\alpha)}(0)$ are the Bernoulli numbers of order α.

We note that

$$C_j(\xi) = B_j^{(j)}(\xi+1), (j \geq 0), \text{(see [23])}. \tag{3}$$

For $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined as

$$e_{\lambda}^{\xi}(z) = (1+\lambda z)^{\frac{\xi}{\lambda}}, e_{\lambda}(z) := e_{\lambda}^{1}(z) = (1+\lambda z)^{\frac{1}{\lambda}}, \text{(see [1–17])}. \tag{4}$$

By (4) and binomial theorem, we have

$$e_{\lambda}^{\xi}(z) = \sum_{\theta=0}^{\infty} (\xi)_{\theta,\lambda}\frac{z^{\theta}}{\theta!}, \tag{5}$$

where $(\xi)_{0,\lambda} = 1, (\xi)_{\theta,\lambda} = \xi(\xi-\lambda)(\xi-2\lambda)\cdots(\xi-(\theta-1)\lambda), (\theta \geq 1)$.

The degenerate Bernoulli polynomials are defined by (see [1,2])

$$\frac{z}{e_{\lambda}(z)-1}e_{\lambda}^{x}(z) = \sum_{j=0}^{\infty} \beta_{j,\lambda}(x)\frac{z^j}{j!}. \tag{6}$$

On putting $x = 0$, $\beta_{j,\lambda} = \beta_{j,\lambda}(0)$ are called the degenerate Bernoulli numbers.

The degenerate Cauchy polynomials $C_{j,\lambda}(\xi)$ are defined by Kim [25] as follows:

$$\begin{aligned}\int_0^1 (1+\log(1+\lambda z)^{\frac{1}{\lambda}})^{\xi+\eta} d\eta &= \frac{\frac{1}{\lambda}(\log(1+\lambda z))}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}(1+\log(1+\lambda z)^{\frac{1}{\lambda}})^{\xi} \\ &= \sum_{j=0}^{\infty} C_{j,\lambda}^{*}(\xi)\frac{z^j}{j!}.\end{aligned} \tag{7}$$

Letting $\xi = 0, C_{j,\lambda} = C_{j,\lambda}(0)$ are the degenerate Cauchy numbers.

In the year 2017, Kim [24] introduced and studied the new class of degenerate Cauchy polynomials $C_{j,\lambda}(\xi)$ of the second kind are given by

$$\frac{z}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}\left(1+\frac{1}{\lambda}\log(1+\lambda z)\right)^{\xi}=\sum_{j=0}^{\infty}C_{j,\lambda}(\xi)\frac{z^j}{j!},\text{(see [16,25,27])}. \tag{8}$$

At the point when $\xi=0, C_{j,\lambda}=C_{j,\lambda}(0)$ are the degenerate Cauchy numbers of the second kind.

The degenerate Daehee polynomials $D_{j,\lambda}(x)$ are defined by (see [6])

$$\frac{\log_{\lambda}(1+z)}{z}(1+z)^{\xi}=\sum_{j=0}^{\infty}D_{j,\lambda}(\xi)\frac{z^j}{j!}. \tag{9}$$

On setting $\xi=0, D_{j,\lambda}=D_{j,\lambda}(0)$ are the degenerate Daehee numbers.

The degenerate Bernoulli polynomials of the second kind are defined by (see [6])

$$\frac{z}{\log_{\lambda}(1+z)}(1+z)^{\xi}=\sum_{j=0}^{\infty}b_{j,\lambda}(\xi)\frac{z^j}{j!}. \tag{10}$$

Letting $\xi=0$, $b_{j,\lambda}=b_{j,\lambda}(0)$ are the degenerate Bernoulli numbers of the second kind.

For $i\geq 0$, the degenerate first kind Stirling numbers are defined by (see [28])

$$\frac{1}{i!}(\log_{\lambda}(1+z))^{i}=\sum_{j=i}^{\infty}S_{1,\lambda}(j,i)\frac{z^j}{j!}. \tag{11}$$

Note that $\lim_{\lambda\to 0}S_{1,\lambda}(j,i)=S_1(j,i)$ are the first kind Stirling numbers given by

$$\frac{1}{i!}(\log(1+z))^{i}=\sum_{j=i}^{\infty}S_{1}(j,i)\frac{z^j}{j!},\quad (i\geq 0),\text{(see [4,26])}. \tag{12}$$

For $i\geq 0$, the degenerate second kind Stirling numbers are given by

$$\frac{1}{i!}(e_{\lambda}(z)-1)^{i}=\sum_{j=i}^{\infty}S_{2,\lambda}(j,i)\frac{z^j}{j!},\text{(see [27])}. \tag{13}$$

We note that $\lim_{\lambda\to 0}S_{2,\lambda}(j,i)=S_2(j,i)$ are the second kind Stirling numbers given by

$$\frac{1}{i!}(e^{z}-1)^{i}=\sum_{j=i}^{\infty}S_{2}(j,i)\frac{z^j}{j!},(i\geq 0),\text{(see [1–29])}. \tag{14}$$

In this paper, Section 3 incorporates the definition of degenerate poly-Cauchy polynomials of the second kind and a preliminary study of these polynomials. Section 4 is a consequence of the definition of the degenerate unipoly-Cauchy polynomials and unipoly polynomials combined with their properties and special cases. Finally, some computational values of degenerate poly-Cauchy polynomials of the second kind are given in Section 5.

2 Degenerate Poly-Cauchy Polynomials and Numbers of the Second Kind

In this segment, we introduce degenerate poly-Cauchy polynomials of the second kind, derived with the aid of modified degenerate polyexponential functions and some identities of these polynomials.

Recently, Kim et al. [4] delivered the modified degenerate polyexponential function defined by

$$\mathrm{Ei}_{k,\lambda}(\xi)=\sum_{j=1}^{\infty}\frac{(1)_{j,\lambda}x^{j}}{(j-1)!j^{k}},(|\xi|<1,k\in\mathbb{Z}). \tag{15}$$

Thus, by

$$\mathrm{Ei}_{1,\lambda}(\xi)=\sum_{j=1}^{\infty}\frac{(1)_{j,\lambda}\xi^{j}}{j!}=e_{\lambda}(\xi)-1. \tag{16}$$

The modified degenerate polyexponential Genocchi (or poly-Genocchi) polynomials are defined by Kim et al. to be (see [7])

$$\frac{\mathrm{Ei}_{k,\lambda}\left(\log_{\lambda}(1+z)\right)}{e_{\lambda}(z)+1}e_{\lambda}^{\xi}(z)=\sum_{j=0}^{\infty}G_{j,\lambda}^{(k)}(\xi)\frac{z^{j}}{j!},(k\in\mathbb{Z}). \tag{17}$$

At the point when $\xi=0$, $G_{j,\lambda}^{(k)}=G_{j,\lambda}^{(k)}(0)$ are the degenerate poly-Genocchi numbers.

By the above definitions, we introduce modified degenerate polyexponential Cauchy (or poly-Cauchy) polynomials of the second kind as

$$\frac{\mathrm{Ei}_{k,\lambda}(\log_{\lambda}(1+z))}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}\left(1+\frac{1}{\lambda}\log(1+\lambda z)\right)^{\xi}=\sum_{j=0}^{\infty}C_{j,\lambda}^{(k)}(\xi)\frac{z^{j}}{j!},(k\in\mathbb{Z}). \tag{18}$$

When $\xi=0$, $C_{j,\lambda}^{(k)}=C_{j,\lambda}^{(k)}(0)$ are the modified degenerate polyexponential Cauchy (or poly-Cauchy) numbers of the second kind.

Theorem 2.1. Let j be non negative number. Then

$$C_{j,\lambda}^{(k)}=\sum_{\sigma=0}^{j}\sum_{\eta=0}^{\sigma}\binom{j}{\sigma}C_{j-\sigma,\lambda}\frac{(1)_{\eta+1,\lambda}S_{1,\lambda}(\sigma+1,\eta+1)}{(\sigma+1)(\eta+1)^{\vartheta-1}}. \tag{19}$$

Proof. Using (11) and (18), we have

$$\sum_{j=0}^{\infty}C_{j,\lambda}^{(k)}\frac{z^{j}}{j!}=\frac{\mathrm{Ei}_{k,\lambda}(\log_{\lambda}(1+z))}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}$$

$$=\frac{z}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}\frac{1}{z}\sum_{\eta=1}^{\infty}\frac{(1)_{\eta,\lambda}(\log_{\lambda}(1+z))^{\eta}}{(\eta-1)!\,\eta^{k}}$$

$$= \frac{z}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}\frac{1}{z}\sum_{\eta=0}^{\infty}\frac{(1)_{\eta+1,\lambda}(\log_{\lambda}(1+z))^{\eta+1}}{\eta!\,(\eta+1)^{k}}\frac{(\eta+1)!}{(\eta+1)!}$$

$$= \frac{z}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}\frac{1}{z}\sum_{\eta=0}^{\infty}\frac{(1)_{\eta+1,\lambda}}{(\eta+1)^{k-1}}\sum_{i=\eta+1}^{\infty}S_{1,\lambda}(i,\eta+1)\frac{z^{i}}{i!}$$

$$= \frac{z}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}\sum_{\eta=0}^{\infty}\frac{(1)_{\eta+1,\lambda}}{(\eta+1)^{k-1}}\sum_{\eta=i}^{\infty}S_{1,\lambda}(i+1,\eta+1)\frac{z^{i}}{(i+1)!}$$

$$= \left(\sum_{j=0}^{\infty}C_{j,\lambda}\frac{z^{j}}{j!}\right)\left(\sum_{i=0}^{\infty}\sum_{\eta=0}^{i}\frac{(1)_{\eta+1,\lambda}}{(\eta+1)^{k-1}}\frac{S_{1,\lambda}(i+1,\eta+1)}{i+1}\frac{z^{i}}{i!}\right)$$

$$= \sum_{j=0}^{\infty}\left(\sum_{\sigma=0}^{j}\sum_{\eta=0}^{\sigma}\binom{j}{\sigma}C_{j-\sigma,\lambda}\frac{(1)_{\eta+1,\lambda}S_{1,\lambda}(\sigma+1,\eta+1)}{(\sigma+1)(\eta+1)^{\vartheta-1}}\right)\frac{z^{j}}{j!}. \tag{20}$$

Therefore, by (15) and (20), we obtain the result (19).

Corollary 2.1. Let j be non negative number. Then

$$C_{j,\lambda}^{(1)} = \sum_{\sigma=0}^{j}\sum_{\eta=0}^{\sigma}\binom{j}{\sigma}C_{j-\sigma,\lambda}\frac{(1)_{\eta+1,\lambda}S_{1,\lambda}(\sigma+1,\eta+1)}{\sigma+1}, (j\geq 0).$$

Theorem 2.2. Let j be non negative number and $k\in\mathbb{Z}$. Then

$$C_{j,\lambda}^{(k)}(\xi) = \sum_{\sigma=0}^{j}\sum_{\rho=0}^{j-\sigma}\binom{j}{\sigma}C_{\sigma,\lambda}^{(k)}(\xi)_{\rho}\lambda^{j-\sigma-\rho}S_{1}(j-\sigma,\rho). \tag{21}$$

Proof. Recall from (18), we have

$$\sum_{j=0}^{\infty}C_{j,\lambda}^{(k)}(\xi)\frac{z^{j}}{j!} = \frac{\mathrm{Ei}_{k,\lambda}(\log_{\lambda}(1+z))}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}\left(1+\frac{1}{\lambda}\log(1+\lambda z)\right)^{\xi}$$

$$= \left(\sum_{\sigma=0}^{\infty}C_{\sigma,\lambda}^{(k)}\frac{z^{\sigma}}{\sigma!}\right)\left(\sum_{\rho=0}^{\infty}\binom{\xi}{\rho}\left(\frac{1}{\lambda}\log(1+\lambda z)\right)^{\rho}\right)$$

$$= \left(\sum_{\sigma=0}^{\infty}C_{\sigma,\lambda}^{(k)}\frac{z^{\sigma}}{\sigma!}\right)\left(\sum_{\rho=0}^{\infty}(\xi)_{\rho}\lambda^{-\rho}\sum_{s=\rho}^{\infty}S_{1}(s,\rho)\lambda^{s}\frac{z^{s}}{s!}\right)$$

$$= \left(\sum_{\sigma=0}^{\infty}C_{\sigma,\lambda}^{(k)}\frac{z^{\sigma}}{\sigma!}\right)\left(\sum_{s=0}^{\infty}\sum_{\rho=0}^{s}(\xi)_{\rho}\lambda^{-\rho}S_{1}(s,\rho)\lambda^{s}\frac{z^{s}}{s!}\right)$$

$$=\sum_{j=0}^{\infty}\left(\sum_{\sigma=0}^{j}\sum_{\rho=0}^{j-\sigma}\binom{j}{\sigma}C_{\sigma,\lambda}^{(k)}(\xi)_{\rho}\lambda^{j-\sigma-\rho}S_1(j-\sigma,\rho)\right)\frac{z^j}{j!}. \tag{22}$$

Thus by (18) and (22), the proof is completed.

Theorem 2.3. Let $j\geq 0$. Then

$$C_{j,\lambda}^{(k)}=\sum_{r=0}^{j}\binom{j}{r}\sum_{r_1+\cdots+r_{k-1}=r}\binom{r}{r_1+\cdots+r_{k-1}}$$
$$\times\frac{b_{r_1,\lambda}(\lambda-1)}{r_1+1}\frac{b_{r_2,\lambda}(\lambda-1)}{r_1+r_2+1}\cdots\frac{b_{r_{k-1},\lambda}(\lambda-1)}{r_1+\cdots+r_{k-1}+1}C_{j-r,\lambda}.$$

Proof. Consider (15), we have

$$\frac{d}{d\xi}\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+\xi))=\frac{d}{d\xi}\sum_{\mathrm{j}=1}^{\infty}\frac{(1)_{\mathrm{j},\lambda}(\log_\lambda(1+\xi))^{\mathrm{j}}}{(\mathrm{j}-1)!\mathrm{j}^{\mathrm{k}}}$$
$$=\frac{(1+\xi)^{\lambda-1}}{\log_\lambda(1+\xi)}\sum_{j=1}^{\infty}\frac{(1)_{j,\lambda}(\log_\lambda(1+\xi))^j}{(j-1)!j^{k-1}}=\frac{(1+x)^{\lambda-1}}{\log_\lambda(1+\xi)}\mathrm{Ei}_{k-1,\lambda}(\log_\lambda(1+\xi)). \tag{23}$$

$$\sum_{j=0}^{\infty}C_{j,\lambda}^{(k)}\frac{\xi^j}{j!}=\frac{1}{\log(1+\frac{1}{\lambda}\log(1+\lambda\xi))}\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+\xi)\right)$$
$$=\frac{1}{\log(1+\frac{1}{\lambda}\log(1+\lambda\xi))}\underbrace{\int_0^{\xi}\frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)}\int_0^{z}\cdots\frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)}\int_0^{z}}_{(k-2)-\text{times}}\frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)}zdz\cdots dz$$
$$=\frac{\xi}{\log(1+\frac{1}{\lambda}\log(1+\lambda\xi))}\sum_{m=0}^{\infty}\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1+\cdots+m_{k-1}}$$
$$\times\frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\frac{b_{m_2,\lambda}(\lambda-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}\frac{\zeta^m}{m!}$$

$$\sum_{j=0}^{\infty}C_{j,\lambda}^{(k)}\frac{\xi^j}{j!}=\sum_{j=0}^{\infty}\sum_{r=0}^{j}\binom{j}{r}\sum_{r_1+\cdots+r_{k-1}=r}\binom{r}{r_1+\cdots+r_{k-1}}$$
$$\times\frac{b_{r_1,\lambda}(\lambda-1)}{r_1+1}\frac{b_{r_2,\lambda}(\lambda-1)}{r_1+r_2+1}\cdots\frac{b_{r_{k-1},\lambda}(\lambda-1)}{r_1+\cdots+r_{k-1}+1}C_{j-r,\lambda}\frac{\xi^j}{j!}, \tag{24}$$

which complete the proof.

Corollary 2.2. Let $j \geq 2$. Then

$$C_{j,\lambda}^{(2)} = \sum_{r=0}^{j} \binom{j}{r} \frac{b_{r,\lambda}(\lambda-1)}{r+1} C_{j-r,\lambda}, (k \geq 2).$$

Theorem 2.4. The following result holds true

$$\tau_{k,\lambda}(-q) = (-1)^q C_{q,\lambda}^{(k)}, \tag{25}$$

$k \geq 1$ and $q \in \mathbb{N} \bigcup \{0\}$, $w \in \mathbb{C}$.

Proof. Let us define the function $\tau_{k,\lambda}(w)$ as

$$\begin{aligned}\tau_{k,\lambda}(w) &= \frac{1}{\Gamma(w)} \int_0^\infty \frac{z^{w-1}}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right) dz \\ &= \frac{1}{\Gamma(w)} \int_0^1 \frac{z^{w-1}}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right) dz \\ &+ \frac{1}{\Gamma(w)} \int_1^\infty \frac{z^{w-1}}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right) dz. \end{aligned} \tag{26}$$

For any $w \in \mathbb{C}$ and absolutely converges, (26) to

$$\lim_{w \to -q} \left| \frac{1}{\Gamma(w)} \int_1^\infty \frac{z^{w-1}}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right) dz \right| \leq \frac{1}{\Gamma(-q)} M = 0. \tag{27}$$

Eq. (27) can be written as

$$\frac{1}{\Gamma(w)} \sum_{l=0}^{\infty} \frac{C_{l,\lambda}^{(k)}}{l!} \frac{1}{w+l}, (\Re(w) > 0)$$

In view of (26) and (27), we have

$$\begin{aligned}\tau_{k,\lambda}(-q) &= \lim_{w \to -q} \frac{1}{\Gamma(q)} \int_0^1 \frac{z^{w-1}}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right) dz \\ &= \lim_{w \to -q} \frac{1}{\Gamma(w)} \int_0^1 z^{w-1} \sum_{l=0}^{\infty} \frac{C_{l,\lambda}^{(k)} z^l}{l!} dz = \lim_{w \to -q} \frac{1}{\Gamma(w)} \sum_{l=0}^{\infty} \frac{C_{l,\lambda}^{(k)}}{w+l} \frac{1}{l!} \\ &= \cdots + 0 + \cdots + 0 + \lim_{w \to -q} \frac{1}{\Gamma(w)} \frac{1}{w+q} \frac{C_{q,\lambda}^{(k)}}{q!} + 0 + 0 + \cdots \\ &= \lim_{w \to -q} \frac{\left(\frac{\Gamma(1-w)\sin \pi w}{\pi}\right)}{w+q} \frac{C_{q,\lambda}^{(k)}}{q!} = \Gamma(1+q)\cos(\pi q) \frac{C_{q,\lambda}^{(k)}}{q!}\end{aligned}$$

$$= (-1)^q C_{q,\lambda}^{(k)}. \tag{28}$$

By (28), we obtain the result.

Theorem 2.5. Let j be non-negative number. Then

$$\sum_{\sigma=0}^{j} \binom{j}{\sigma} C_{j-\sigma}(\xi) \frac{\lambda^{-\sigma-1}(1)_{\sigma+1,\lambda}}{(\sigma+1)^k} = \sum_{\sigma=0}^{j} \lambda^{j-\sigma} C_{\sigma,\lambda}^{(k)}(\xi) S_2(j,\sigma).$$

Proof. By changing z with $\frac{1}{\lambda}e_\lambda(z) - 1$ in (18) that

$$\begin{aligned}
\sum_{i=0}^{\infty} C_{i,\lambda}^{(k)}(\xi)\lambda^{-i}\frac{(e_\lambda(z)-1)^i}{i!} &= \frac{\mathrm{Ei}_{k,\lambda}(\frac{z}{\lambda})}{\log(1+z)}(1+z)^\xi \\
&= \left(\frac{z}{\log(1+z)}(1+z)^\xi\right)\left(\frac{\mathrm{Ei}_{k,\lambda}(\frac{z}{\lambda})}{z}\right) \\
&= \left(\sum_{j=0}^{\infty} C_j(\xi)\frac{z^j}{j!}\right)\left(\frac{1}{z}\sum_{i=1}^{\infty}\frac{\lambda^{-i}(1)_{i,\lambda}z^i}{(i-1)!\,i^k}\right) \\
&= \left(\sum_{j=0}^{\infty} C_j(\xi)\frac{z^j}{j!}\right)\left(\sum_{i=0}^{\infty}\frac{\lambda^{-i-1}(1)_{i+1,\lambda}z^i}{i!\,(i+1)^k}\right) \\
&= \sum_{j=0}^{\infty}\left(\sum_{\sigma=0}^{j}\binom{j}{\sigma}C_{j-\sigma}(\xi)\frac{\lambda^{-\sigma-1}(1)_{\sigma+1,\lambda}}{(\sigma+1)^k}\right)\frac{z^j}{j!}. \qquad (29)
\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}
\sum_{i=0}^{\infty} C_{i,\lambda}^{(k)}(x)\lambda^{-i}\frac{(e_\lambda(z)-1)^i}{i!} &= \sum_{i=0}^{\infty} C_{i,\lambda}^{(k)}(\xi)\lambda^{-i}\sum_{j=i}^{\infty} S_2(j,i)\lambda^j\frac{z^j}{j!} \\
&= \sum_{j=0}^{\infty}\left(\sum_{\sigma=0}^{j}\lambda^{j-\sigma}C_{\sigma,\lambda}^{(k)}(\xi)S_2(j,\sigma)\right)\frac{z^j}{j!}. \qquad (30)
\end{aligned}$$

In view of (29) and (30), we obtain the result.

Theorem 2.6. Let η be non-negative number. Then

$$\sum_{\sigma=1}^{\eta}\frac{(1)_{\sigma,\lambda}S_{1,\lambda}(\eta,\sigma)}{\sigma^{k-1}}$$

$$= \sum_{\sigma=1}^{\eta}\sum_{\rho=1}^{\sigma}\binom{\eta}{\sigma}C_{\eta-\sigma,\lambda}^{(k)}(\rho-1)!\,(-1)^{\rho-1}\lambda^{\sigma-\rho}S_1(\sigma,\rho). \tag{31}$$

Proof. Consider the Eq. (18), we have

$$\mathrm{Ei}_{k,\lambda}(\log_{\lambda}(1+z)) = \sum_{\eta=1}^{\infty}\left(\sum_{\sigma=1}^{\eta}\frac{(1)_{\sigma,\lambda}S_{1,\lambda}(\eta,\sigma)}{\sigma^{k-1}}\right)\frac{z^{\eta}}{\eta!}. \tag{32}$$

$$= \left(\sum_{\eta=0}^{\infty}C_{\eta,\lambda}^{(k)}\frac{z^{\eta}}{\eta!}\right)\left(\log(1+\frac{1}{\lambda}\log(1+\lambda z))\right)$$

$$= \left(\sum_{\eta=0}^{\infty}C_{\eta,\lambda}^{(k)}\frac{z^{\eta}}{\eta!}\right)\left(\log(1+\frac{1}{\lambda}\log(1+\lambda z))\right)$$

$$= \left(\sum_{\eta=0}^{\infty}C_{\eta,\lambda}^{(k)}\frac{z^{\eta}}{\eta!}\right)\left(\sum_{\rho=1}^{\infty}\frac{(-1)^{\rho-1}}{\rho}\lambda^{-\rho}(\log(1+\lambda z))^{\rho}\right)$$

$$= \left(\sum_{\eta=0}^{\infty}C_{\eta,\lambda}^{(k)}\frac{z^{\eta}}{\eta!}\right)\left(\sum_{\rho=1}^{\infty}(\rho-1)!\,(-1)^{\rho-1}\lambda^{-\rho}\sum_{\sigma=\rho}^{\infty}S_1(\sigma,\rho)\lambda^{\sigma}\frac{z^{\sigma}}{\sigma!}\right)$$

$$= \left(\sum_{\eta=0}^{\infty}C_{\eta,\lambda}^{(k)}\frac{z^{\eta}}{\eta!}\right)\left(\sum_{\sigma=1}^{\infty}\left(\sum_{\rho=1}^{\sigma}(\rho-1)!\,(-1)^{\rho-1}\lambda^{\sigma-\rho}S_1(\sigma,\rho)\right)\frac{z^{\sigma}}{\sigma!}\right)$$

$$= \sum_{\eta=1}^{\infty}\left(\sum_{\sigma=1}^{\eta}\sum_{\rho=1}^{\sigma}\binom{\eta}{\sigma}C_{\eta-\sigma,\lambda}^{(k)}(\rho-1)!\,(-1)^{\rho-1}\lambda^{\sigma-\rho}S_1(\sigma,\rho)\right)\frac{z^{\eta}}{\eta!}. \tag{33}$$

The complete of the Proof.

Corollary 2.3. Let η be non-negative number, we have

$$\sum_{\sigma=1}^{\eta}(1)_{\sigma,\lambda}S_{1,\lambda}(\eta,\sigma) = \sum_{\sigma=1}^{\eta}\sum_{\rho=1}^{\sigma}\binom{\eta}{\sigma}C_{\eta-\sigma,\lambda}^{(1)}(\rho-1)!\,(-1)^{\rho-1}\lambda^{\sigma-\rho}S_1(\sigma,\rho).$$

Theorem 2.7. Let j be non-negative number. Then

$$C_{j,\lambda}^{(k)} = \sum_{s=0}^{j}\binom{j}{s}\sum_{r=0}^{s}\binom{s}{r}\sum_{q=0}^{r}\frac{(1)_{q+1,\lambda}S_{1,\lambda}(r+1,q+1)}{(r+1)(q+1)^{k-1}}C_{s-r,\lambda}^{*}\lambda^{j-s}B_{j-s}^{(j-s)}.$$

Proof. We observe that

$$\frac{\mathrm{Ei}_{k,\lambda}(\log_{\lambda}(1+z))}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} = \left(\frac{\lambda z}{\log(1+\lambda z)}\right)\left(\frac{\mathrm{Ei}_{k,\lambda}(\log_{\lambda}(1+z))}{z}\right)\left(\frac{\frac{1}{\lambda}\log(1+\lambda z)}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}\right)$$

$$= \left(\sum_{j=0}^{\infty}\lambda^{l}B_{j}^{(j)}\frac{z^{j}}{j!}\right)\left(\frac{1}{z}\sum_{q=1}^{\infty}\frac{(1)_{q,\lambda}(\log_{\lambda}(1+z))^{q}}{(q-1)!\,q^{k}}\right)\left(\sum_{l=0}^{\infty}C_{l,\lambda}^{*}\frac{z^{l}}{l!}\right)$$

$$= \left(\sum_{j=0}^{\infty} \lambda^l B_j^{(j)} \frac{z^j}{j!}\right) \left(\frac{1}{z}\sum_{q=0}^{\infty} \frac{(1)_{q+1,\lambda}(\log_\lambda(1+z))^{q+1}}{q!\,(q+1)^k} \frac{(q+1)!}{(q+1)!}\right) \left(\sum_{l=0}^{\infty} C_{l,\lambda}^{*} \frac{z^l}{l!}\right)$$

$$= \left(\sum_{j=0}^{\infty} \lambda^l B_j^{(j)} \frac{z^j}{j!}\right) \left(\frac{1}{z}\sum_{q=0}^{\infty} \frac{(1)_{q+1,\lambda}}{(q+1)^{k-1}} \sum_{r=q+1}^{\infty} S_{1,\lambda}(r,q+1)\frac{z^r}{r!}\right) \left(\sum_{l=0}^{\infty} C_{l,\lambda}^{*} \frac{z^l}{l!}\right)$$

$$= \left(\sum_{j=0}^{\infty} \lambda^j B_j^{(j)} \frac{z^j}{j!}\right) \left(\sum_{r=0}^{\infty}\sum_{q=0}^{r} \frac{(1)_{q+1,\lambda} S_{1,\lambda}(r+1,q+1)}{(r+1)(q+1)^{k-1}} \frac{z^r}{r!}\right) \left(\sum_{s=0}^{\infty} C_{s,\lambda}^{*} \frac{z^s}{s!}\right)$$

$$= \left(\sum_{j=0}^{\infty} \lambda^j B_j^{(j)} \frac{z^j}{j!}\right) \left(\sum_{l=0}^{\infty}\sum_{r=0}^{l} \binom{l}{r} \sum_{q=0}^{r} \frac{(1)_{q+1,\lambda} S_{1,\lambda}(r+1,q+1)}{(r+1)(q+1)^{k-1}} C_{s-r,\lambda}^{*}\right) \frac{z^s}{s!}$$

$$= \sum_{j=0}^{\infty} \left(\sum_{s=0}^{j} \binom{j}{l} \sum_{r=0}^{s} \binom{s}{r} \sum_{q=0}^{r} \frac{(1)_{q+1,\lambda} S_{1,\lambda}(r+1,q+1)}{(r+1)(q+1)^{k-1}} C_{s-r,\lambda}^{*} \lambda^{j-l} B_{j-q}^{(j-q)}\right) \frac{z^j}{j!}. \tag{34}$$

Therefore, by (15) and (34), we acquire the desired result.

Theorem 2.8. Let j be non-negative number. Then

$$\frac{C_{j+1,\lambda}^{(k)}(1) - C_{j+1,\lambda}^{(k)}}{j+1} = \sum_{r=0}^{j} \binom{j}{r} \sum_{\eta=0}^{r} \frac{(1)_{\eta+1,\lambda} S_{1,\lambda}(r+1,\eta+1)}{(r+1)(\eta+1)^{k-1}} C_{j-r,\lambda}^{*}.$$

Proof. Consider the following expression:

$$\sum_{j=0}^{\infty} C_{j,\lambda}^{(k)}(1)\frac{z^j}{j!} = \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+z))}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \left(1+\frac{1}{\lambda}\log(1+\lambda z)\right). \tag{35}$$

$$= \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+z))}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} + \frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+z))\frac{1}{\lambda}\log(1+\lambda z)}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}$$

$$= \sum_{j=0}^{\infty} C_{j,\lambda}^{(k)}\frac{z^j}{j!} + z\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+z))}{z} \frac{\frac{1}{\lambda}\log(1+\lambda z)}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}$$

$$\sum_{j=1}^{\infty} \left[C_{j,\lambda}^{(k)}(1) - C_{j,\lambda}^{(k)}\right]\frac{z^j}{j!} = z\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+z))}{z} \frac{\frac{1}{\lambda}\log(1+\lambda z)}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}$$

$$= \left(\frac{1}{z}\sum_{\eta=1}^{\infty} \frac{(1)_{\eta,\lambda}(\log_\lambda(1+z))^{\eta}}{(\eta-1)!\,\eta^k}\right) \left(\sum_{l=0}^{\infty} C_{l,\lambda}^{*}\frac{z^l}{l!}\right)$$

$$= \left(\sum_{\eta=0}^{\infty} \frac{(1)_{\eta+1,\lambda} (\log_\lambda(1+z))^{\eta+1}}{\eta!\,(\eta+1)^k} \frac{(\eta+1)!}{(\eta+1)!} \right) \left(\sum_{j=0}^{\infty} C^*_{j,\lambda} \frac{z^j}{j!} \right)$$

$$= \left(\sum_{r=0}^{\infty} \sum_{\eta=0}^{r} \frac{(1)_{\eta+1,\lambda} S_{1,\lambda}(r+1,\eta+1)}{(r+1)(\eta+1)^{k-1}} \frac{z^r}{r!} \right) \left(\sum_{j=0}^{\infty} C^*_{j,\lambda} \frac{z^j}{j!} \right)$$

$$= \sum_{j=0}^{\infty} \left(\sum_{r=0}^{j} \binom{j}{r} \sum_{\eta=0}^{r} \frac{(1)_{\eta+1,\lambda} S_{1,\lambda}(r+1,\eta+1)}{(r+1)(\eta+1)^{k-1}} C^*_{j-r,\lambda} \right) \frac{z^j}{j!}. \tag{36}$$

From Eq. (35), we have

$$\sum_{j=1}^{\infty} \left[C^{(k)}_{j,\lambda}(1) - C^{(k)}_{j,\lambda} \right] \frac{t^{j-1}}{j!}$$

$$= \sum_{j=0}^{\infty} \left[\frac{C^{(k)}_{j+1,\lambda}(1) - C^{(k)}_{j+1,\lambda}}{j+1} \right] \frac{z^j}{j!}. \tag{37}$$

Thus, by (36) and (37), we complete the proof.

3 DegenerAte Unipoly-Cauchy Polynomials of the Second Kind

In this section, we introduce degenerate unipoly-Cauchy polynomials of the second kind by using degenerate unipoly function and derive the relationships between degenerate Daehee polynomials and degenerate Cauchy polynomials of the second kind.

In [25], Dolgy and Khan introduced degenerate unipoly function given by

$$u_{k,\lambda}(\xi|p) = \sum_{j=1}^{\infty} p(j) \frac{(1)_{j,\lambda} \xi^j}{j^k} \tag{38}$$

Note that, we have

$$u_{k,\lambda}\left(\xi | \frac{1}{\Gamma}\right) = \mathrm{Ei}_{k,\lambda}(\xi) \tag{39}$$

is the modified degenerate polyexponential function, where $\Gamma := \Gamma(j)$ is well-known as Gamma function.

It is clear that

$$\lim_{\lambda\to 0} u_{k,\lambda}(\xi|p) = \sum_{j=1}^{\infty} \lim_{\lambda\to 0} p(j) \frac{(1)_{j,\lambda} \xi^j}{j^k}$$

$$= u_k(\xi|p) = \sum_{j=1}^{\infty} p(j) \frac{\xi^j}{j^k}, (k \in \mathbb{Z}) \tag{40}$$

are called the unipoly function attached to polynomials $p(x)$ (see [3]).

From (40), we have

$$u_k(\xi|1)=\sum_{j=1}^{\infty}\frac{\xi^j}{j^k}=\mathrm{Li}_k(\xi),(\text{see [26]}), \tag{41}$$

is the ordinary polylogarithm function.

By using (15) and (38), the degenerate unipoly-Cauchy polynomials of the second kind is given by the following generating function

$$\frac{u_{k,\lambda}(\log_\lambda(1+z)|p)}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}\left(1+\frac{1}{\lambda}\log(1+\lambda z)\right)^{\xi}=\sum_{j=0}^{\infty}C_{j,\lambda,p}^{(k)}(\xi)\frac{z^j}{j!}. \tag{42}$$

In the case when $\xi=0, C_{j,\lambda,p}^{(k)}=C_{j,\lambda,p}^{(k)}(0)$ are the degenerate unipoly-Cauchy numbers of the second kind.

Theorem 3.1. Let $j\geq 0$. Then

$$C_{j,\lambda,\frac{1}{\Gamma}}^{(k)}(\xi)=C_{j,\lambda}^{(k)}(\xi).$$

Proof. On taking $p(j)=\frac{1}{\Gamma\lambda}$. Then we have

$$\begin{aligned}\sum_{j=0}^{\infty}C_{j,\lambda,\frac{1}{\Gamma}}^{(k)}(\xi)\frac{z^j}{j!}&=\frac{u_{k,\lambda}(\log_\lambda(1+z)|\frac{1}{\Gamma}p)}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}\left(1+\frac{1}{\lambda}\log(1+\lambda z)\right)^{\xi}\\&=\frac{1}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}(\log_\lambda(1+z))^m}{m^k(m-1)!}\left(1+\frac{1}{\lambda}\log(1+\lambda z)\right)^{\xi}\\&=\frac{\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+z))}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}\left(1+\frac{1}{\lambda}\log(1+\lambda z)\right)^{\xi}\\&=\sum_{j=0}^{\infty}C_{j,\lambda}^{(k)}(\xi)\frac{z^j}{j!}.\end{aligned} \tag{43}$$

In view of (43), we obtain the result.

Theorem 3.2. Let j be non-negative number. Then

$$C_{j,\lambda,p}^{(k)}=\sum_{l=0}^{\infty}\sum_{\eta=0}^{\zeta}\binom{j}{\zeta}\frac{p(\eta+1)(1)_{\eta+1,\lambda}(\eta+1)!\,S_{1,\lambda}(\zeta+1,\eta+1)C_{j-\zeta,\lambda}}{(\eta+1)^k(\zeta+1)}. \tag{44}$$

Proof. Consider the Eq. (42), we have

$$\sum_{j=0}^{\infty} C_{j,\lambda,p}^{(k)} \frac{z^j}{j!} = \frac{u_{k,\lambda}(\log_\lambda(1+z)|p)}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}$$
$$= \frac{1}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \sum_{\eta=1}^{\infty} \frac{p(\eta)(1)_{\eta,\lambda}}{\eta^k} (\log_\lambda(1+z))^\eta$$
$$= \frac{1}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \sum_{\eta=0}^{\infty} \frac{p(\eta+1)(1)_{\eta+1,\lambda}(\eta+1)!}{(\eta+1)^k} \sum_{\zeta=\eta+1}^{\infty} S_{1,\lambda}(\eta+1,\zeta) \frac{z^\zeta}{\zeta!}$$
$$= \left(\sum_{j=0}^{\infty} C_{j,\lambda} \frac{z^j}{j!}\right) \left(\sum_{\eta=0}^{\infty} \sum_{\zeta=0}^{\eta} \frac{p(\eta+1)(1)_{\eta+1,\lambda}(\eta+1)!}{(\eta+1)^k} S_{1,\lambda}(\zeta,\eta+1) \frac{z^\zeta}{\zeta!}\right)$$
$$= \sum_{j=0}^{\infty} \left(\sum_{\zeta=0}^{\infty} \sum_{\eta=0}^{\zeta} \binom{j}{\zeta} \frac{p(\eta+1)(1)_{\eta+1,\lambda}(\eta+1)!\, S_{1,\lambda}(\zeta+1,\eta+1) C_{j-\zeta,\lambda}}{(\eta+1)^k(\zeta+1)}\right) \frac{z^j}{j!}. \tag{45}$$

By (42) and (45), we complete the proof.

Corollary 3.1. Let $j \geq 0$. Then

$$C_{j,\lambda,\frac{1}{\Gamma}}^{(k)} = C_{j,\lambda}^{(k)} = \sum_{\zeta=0}^{\infty} \sum_{\eta=0}^{\zeta} \binom{j}{l} \frac{S_{1,\lambda}(j+1,\zeta+1) C_{j-\zeta,\lambda}}{(j+1)^{k-1}(\zeta+1)}.$$

Theorem 3.3. Let $j \geq 0$. Then

$$C_{j,\lambda,p}^{(k)}(\xi) = \sum_{\zeta=0}^{j} \sum_{\eta=0}^{j-\zeta} C_{\zeta,\lambda,p}^{(k)}(\xi)_\eta \lambda^{s-\eta} S_1(j-\zeta,\eta). \tag{46}$$

Proof. Recall from (42), we see that

$$\sum_{j=0}^{\infty} C_{j,\lambda}^{(k,p)}(\xi) \frac{z^j}{j!} = \frac{u_{k,\lambda}(\log_\lambda(1+z)|p)}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \left(1+\frac{1}{\lambda}\log(1+\lambda z)\right)^\xi$$
$$= \frac{u_{k,\lambda}(\log_\lambda(1+z)|p)}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \sum_{\eta=0}^{\infty} \binom{\xi}{\eta} \left(\frac{1}{\lambda}\log(1+\lambda z)\right)^\eta$$
$$= \left(\sum_{\zeta=0}^{\infty} C_{\zeta,\lambda,p}^{(k)} \frac{z^\zeta}{\zeta!}\right) \left(\sum_{\eta=0}^{\infty} (\xi)_\eta \lambda^{s-\eta} \sum_{s=\eta}^{\infty} S_1(s,\eta) \frac{z^s}{s!}\right)$$

$$= \left(\sum_{\zeta=0}^{\infty} C_{\zeta,\lambda,p}^{(k)} \frac{z^{\zeta}}{\zeta!}\right) \left(\sum_{s=0}^{\infty} \sum_{\eta=0}^{s} (\xi)_{\eta} \lambda^{s-\eta} S_1(s,\eta) \frac{z^s}{s!}\right)$$

$$= \sum_{j=0}^{\infty} \left(\sum_{\zeta=0}^{j} \sum_{\eta=0}^{j-\zeta} C_{\zeta,\lambda,p}^{(k)} (\xi)_{\eta} \lambda^{s-\eta} S_1(j-\zeta,\eta)\right) \frac{z^j}{j!}. \tag{47}$$

Thus, by (47), we get the desired result.

Theorem 3.4. Let $j \geq 0$. Then

$$C_{j,\lambda,p}^{(k)} = \sum_{\zeta=0}^{j} \sum_{a=0}^{j-\zeta} \sum_{\eta=0}^{\zeta} \binom{j}{\zeta} \binom{j}{a} D_{j-\zeta-a,\lambda} C_{a,\lambda} \frac{p(\eta+1)(1)_{\eta+1,\lambda}(\eta+1)!}{(\eta+1)^k} S_{1,\lambda}(\zeta,\eta+1). \tag{48}$$

Proof. Using (42), we have

$$\sum_{j=0}^{\infty} C_{j,\lambda,p}^{(k)} \frac{z^j}{j!} = \frac{u_{k,\lambda}(\log_{\lambda}(1+z)|p)}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))}$$

$$= \frac{1}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \sum_{\eta=1}^{\infty} \frac{p(\eta)(1)_{\eta,\lambda}}{\eta^k} (\log_{\lambda}(1+z))^{\eta+1}$$

$$= \frac{1}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \sum_{\eta=0}^{\infty} \frac{p(\eta+1)(1)_{\eta+1,\lambda}}{(\eta+1)^k} (\log_{\lambda}(1+z))^{\eta+1}$$

$$= \frac{\log_{\lambda}(1+z)}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \sum_{\eta=0}^{\infty} \frac{p(\eta+1)(1)_{\eta+1,\lambda}(\eta+1)!}{(\eta+1)^k(\eta+1)!} (\log_{\lambda}(1+z))^{\eta+1}$$

$$= \frac{\log_{\lambda}(1+z)}{z} \frac{z}{\log(1+\frac{1}{\lambda}\log(1+\lambda z))} \sum_{\eta=0}^{\infty} \frac{p(\eta+1)(1)_{\eta+1,\lambda}(\eta+1)!}{(\eta+1)^k} \sum_{\zeta=\eta}^{\infty} S_{1,\lambda}(\zeta,\eta+1) \frac{z^{\zeta}}{\zeta!}$$

$$= \left(\sum_{s=0}^{\infty} D_{s,\lambda} \frac{z^s}{s!}\right) \left(\sum_{a=0}^{\infty} C_{a,\lambda} \frac{z^a}{a!}\right) \left(\sum_{\zeta=0}^{\infty} \sum_{\eta=0}^{\zeta} \frac{p(\eta+1)(1)_{\eta+1,\lambda}(\eta+1)!}{(\eta+1)^k} S_{1,\lambda}(\zeta,\eta+1) \frac{z^{\zeta}}{\zeta!}\right)$$

$$= \left(\sum_{b=0}^{\infty} \sum_{a=0}^{b} \binom{b}{a} D_{b-a,\lambda} C_{a,\lambda} \frac{z^b}{b!}\right) \left(\sum_{\zeta=0}^{\infty} \sum_{\eta=0}^{\zeta} \frac{p(\eta+1)(1)_{\eta+1,\lambda}(\eta+1)!}{(\eta+1)^k} S_{1,\lambda}(\zeta,\eta+1) \frac{z^{\zeta}}{\zeta!}\right)$$

$$= \sum_{j=0}^{\infty} \left(\sum_{\zeta=0}^{j} \sum_{a=0}^{j-\zeta} \sum_{\eta=0}^{\zeta} \binom{j}{\zeta} \binom{j}{a} D_{j-\zeta-a,\lambda} C_{a,\lambda} \frac{p(\eta+1)(1)_{\eta+1,\lambda}(\eta+1)!}{(\eta+1)^k} S_{1,\lambda}(\zeta,\eta+1)\right) \frac{z^j}{j!}. \tag{49}$$

In view of (49), we complete the proof.

4 Computational Values and Graphical Representation of Degenerate Poly-Cauchy Polynomials of the Second Kind

In this section, sure numerical computations are carried out to calculate sure contributors of the degenerate poly-Cauchy polynomials of the second kind and display some graphical representations. The first six individuals of $C_{j,\lambda}^{(k)}(\xi)$ are calculated and given as

$$C_{0,\lambda}^{(k)}(\xi) = 1,$$

$$C_{1,\lambda}^{(k)}(\xi) = \frac{159}{32} + \xi$$

$$C_{2,\lambda}^{(k)}(\xi) = \frac{43723}{3888} + \frac{63\xi}{16} + \xi^2$$

$$C_{3,\lambda}^{(k)}(\xi) = \frac{844553}{18432} + \frac{3661\xi}{324} - \frac{99\xi^2}{32} + \xi^3$$

$$C_{4,\lambda}^{(k)}(\xi) = -\frac{1953476014259}{9720000000} + \frac{263515\xi}{13824} + \frac{55549\xi^2}{648} - \frac{129\xi^3}{8} + \xi^4$$

$$C_{5,\lambda}^{(k)}(\xi) = \frac{17044747582391}{3888000000} - \frac{281099\xi}{9000000} - \frac{46995065\xi^2}{27648} + \frac{416095\xi^3}{972} - \frac{1125\xi^4}{32} + \xi^5$$

To show the behavior of $C_{j,\lambda}^{(k)}(\xi)$, we display the graph of $C_{j,\lambda}^{(k)}(\xi)$ for $k = 7$ and $\lambda = 5$, this graph is presented in Fig. 1.

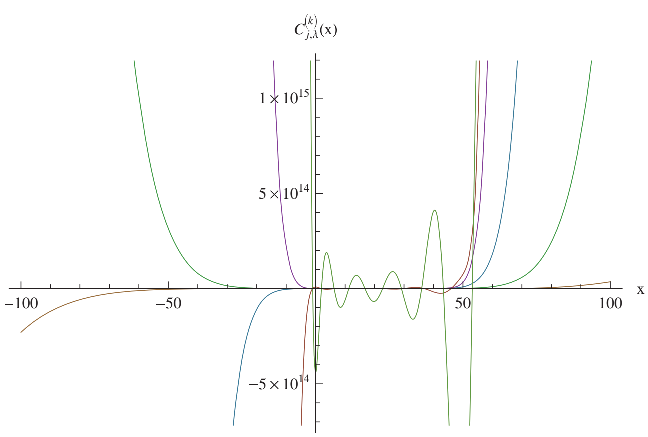

Figure 1: Graph of $C_{j,\lambda}^{(k)}(\xi),\ j = 1, 2, \ldots, 12$

5 Conclusions

In this paper, we have presented the degenerate poly-Cauchy numbers and polynomials of the second kind and discussed, in particular, some interesting series representations. We have deduced some relevant properties by using the structure and the relations satisfied by the recently degenerate polyexponential functions. Section 3 incorporates the definition of degenerate poly-Cauchy polynomials of the second kind and a preliminary study of these polynomials. Section 4 is a consequence of the definition of the degenerate unipoly-Cauchy polynomials and unipoly polynomials combined with their properties and special cases. Finally, some computational values of degenerate poly-Cauchy polynomials of the second kind are given in Section 5.

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