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ANALYTICAL SOLUTION FOR A CLASS OF FLAT PLATE CONJUGATE CONVECTIVE HEAT TRANSFER PROBLEMS

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ABSTRACT

Analytical solutions for three different flat plate conjugate heat transfer cases are presented. The cases are as follows:transient heat transfer of a thin plate with uniform heat generation; the Luikov problem in which one plate surface is kept in a constant temperature and the other one is cooled by forced convection ; and a modified Luikov problem with heat generation on one surface and convection on both surfaces of the plate. All the cases are solved for both laminar and turbulent flows with $Pr \gtrsim 1$. The solutions in the paper are based on the superposition principle and analytical expressions are used to couple the temperature and the heat flux distributions on the surface of the plate. The results of the Luikov problem and transient plate are also compared to other solutions presented earlier in the literature.
Keywords: *flat plate, conjugate heat transfer, Luikov problem, fractional calculus*

1. INTRODUCTION

Convective heat transfer problems are difficult to solve even for fixed boundary conditions. The solution process is even more complicated when conduction or thermal energy storage has to be taken into account. In these conjugated cases, analytical solutions are usually obtained only by making rather rough approximations such as using a constant convective heat transfer coefficient. For turbulent flows this type of assumption is often adequate but for laminar flows this may give rise to to significant errors.

Conjugate heat transfer, in which conduction in a plate and convection from the surfaces are coupled together, has been discussed in a number of papers. The concept of conjugated heat transfer was introduced by Perelmann (1961). A very important problem in the field was formulated by Luikov, who presented the solution when one of the plate surfaces has a fixed constant temperature and the other one was cooled by forced laminar flow (Luikov, 1974). Subsequently, flat plate problems have been treated in many papers (Payvar, 1977; Karvinen, 1978a; Pozzi and Lupo, 1989; Pop and Ingham, 1993; Treviño *et al.*, 1997; Treviño and Liñán, 1984; Mosaad, 1999). Two-dimensional analyses, in which the effect of longitudinal and transversal conduction are included are also found in the literature (Vynnycki *et al.*, 1998; Chida, 2000).

The reason for the present study was to understand the cooling problem of an electroluminescence display, which consists of a plate, on the other side of which a thin active film of different material is processed. After the manufacturing process the display must be aged in order to obtain the desired properties for the display. In the ageing process a uniform heat flux greater than that in actual use is generated in an active layer and the plate has to be cooled by forced convection from both surfaces in order to prevent over-heating.

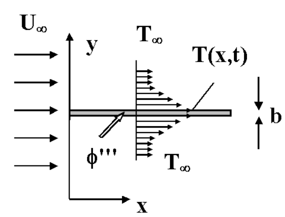

Fig. 1 Thin plate with uniform heat generation and convectively cooled from both surfaces.

It became evident that a similar kind of solution method as used to solve the ageing problem of a display can also be applied to other types of conjugate problems. Thus, analytical solutions of a transient thin flat plate problem and the treatment of classical Luikov problem are included in the paper.

The first specific problem discussed is transient heat transfer of a thin plate shown in Fig. 1. The goal is to find the transient temperature of a thin plate cooled by forced convection, when there is a step change of a spatially uniform heat generation in the plate. The problem has been solved previously by one of the authors both numerically and experimentally (Karvinen, 1978b). However, to the authors knowledge no analytical solution of the problem has earlier been presented.

The second problem to be solved is the classical Luikov problem shown in Fig. 2. The goal is to find the temperature difference across a plate when one surface is kept at a uniform temperature and the other surface is cooled by forced convection.

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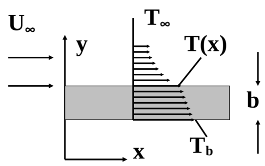

Fig. 2 Plate with uniform temperature T_b at lower surface and convectively cooled at upper surface (Luikov problem).

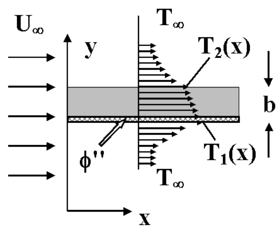

Fig. 3 A plate with uniform heat generation ϕ'' at lower surface and convection cooling at both surfaces.

The third problem is the electroluminescence display problem already discussed briefly above and shown in Fig. 3. The lower surface of the plate is subject to a constant heat flux and the plate is cooled by forced convection from both surfaces. No solution of this problem has previously been published.

The conjugated problems above have usually been solved by starting from the governing partial equations. In this paper a different type of approach is adopted. Due to the linearity of the convection problem with respect to the temperature boundary condition, the superposition principle can be used. The heat flux distribution at the plate surface and the surface temperature can be coupled very acccurately together with an integral equation.

2. PLATE SURFACE TEMPERATURE WITH ARBITRARY HEAT FLUX

The surface temperature $T(x)$ for a prescribed surface heat flux distribution $q(x)$ can be expressed as

$$T(x) - T_\infty = \frac{Re_x^{-m} Pr^{-n}}{C k_f} \int_0^x \left[1 - \left(\frac{s}{x}\right)^\gamma\right]^{\beta-1} q(s) ds \qquad (1)$$

Equation (1) is obtained using an integral method for a constant surface temperature plate with an unheated starting length, which can be extended for any arbitrary variation in surface temperature or heat flux by applying the method of superposition. When deriving Eq. (1) some approximations have been done. In the case of a laminar boundary layer it is assumed that the hydrodynamic boundary layer is thicker than the thermal boundary layer, which means that the Prandtl number is close to 1. In the case of a turbulent boundary layer the derivation of Eq. (1) is even more sophisticated including for instance the assumption that the turbulent Prandtl number is 1 (see e.g. Kays *et al.* (2005)). In spite of approximations the accuracy of Eq. (1) is very good and it is valid for the both laminar and turbulent boundary layers when $Pr \gtrsim 1$. The constants C, m, n, γ and β are given in Table 1 for laminar and turbulent flows.

3. TRANSIENT HEAT TRANSFER OF THIN PLATE

For a thin plate in Fig. 1, the temperature distribution in the plate is constant in y-dirction. At time $t = 0$ the temperature of the plate equals that of the free stream. For $t > 0$ there is spatially uniform heat generation ϕ''' in the plate. It is well-known that in actual practice the time constant

Table 1 Constants in Eq. (1) for laminar and turbulent flows

Flow type	C	γ	β	m	n
Laminar	1.605	3/4	1/3	1/2	1/3
Turbulent	0.293	9/10	1/9	4/5	3/5

of the thermal boundary layer is much smaller than that of the wall, the order of magnitude being seconds for step change of surface temperature. In that case, the problem can be as a quasi-static, i.e. convection is steady-state but depends on the surface temperature. Thus, neglecting streamwise conduction in the plate, the energy balance yields

$$2q(x,t) = \phi''' b - \rho c_p b \frac{\partial T(x,t)}{\partial t} \qquad (2)$$

The solution is sought for the dimensionless variable

$$\theta_{ns}(x,t) = \frac{T(x,t) - T_\infty}{T_{ss}(x) - T_\infty} \qquad (3)$$

where

$$T_{ss}(x) - T_\infty = \frac{B(\beta, 1/\gamma) x \phi''' b}{2 C k_f \gamma Pr^n Re_x^m} \qquad (4)$$

is the steady state temperature distribution of the thin plate. Substituting Eqs. (2)–(4) into Eq. (1) and performing the change of variables

$$\tau = \left(\frac{k_f Re_x^m Pr^n t}{x \rho c_p b}\right)^{\frac{\gamma}{m-1}} \qquad (5)$$

and

$$\frac{\tau'}{\tau} = \left(\frac{s}{x}\right)^\gamma \qquad (6)$$

after algebraic calculation, yields

$$\theta_{ns}(\tau) = 1 - \frac{\tau^{1-\beta-\frac{1}{\gamma}}}{C(m-1)} \int_0^\tau (\tau - \tau')^{\beta-1} \tau'^{\frac{2-m}{\gamma}} \frac{d\theta_{ns}}{d\tau'} d\tau' \qquad (7)$$

Using the Riemann-Liouville fractional integral operator D, defined in the appendix by Eq. (35), the integro-differential equation (7) can be rewritten as

$$\theta_{ns}(\tau) = 1 - \frac{\Gamma(\beta) \tau^{1-\beta-\frac{1}{\gamma}}}{C(m-1)} D_\tau^{-\beta} \left(\tau^{\frac{2-m}{\gamma}} \frac{d\theta_{ns}}{d\tau}\right) \qquad (8)$$

Next, the generalised Leibniz rule given by Eq. (37) is applied to Eq. (8). It yields an ordinary differential equation of infinite order

$$\theta_{ns}(\tau) = 1 - \frac{\Gamma(\beta) \tau^{1-\beta-\frac{1}{\gamma}}}{C(m-1)} \sum_{i=0}^{\infty} \binom{-\beta}{i} \left(D_\tau^{-\beta-i} \tau^{\frac{2-m}{\gamma}}\right) \frac{d^{i+1}\theta_{ns}}{d\tau^{i+1}} \qquad (9)$$

Performing the change of variables

$$K = \frac{k_f Re_x^m Pr^n t}{x \rho c_p b} = \tau^{\frac{m-1}{\gamma}}$$

and using the chain rule for $(i+1)$th order differentiation of a composite function (see appendix, Eq. (39)) finally yields

$$\theta_{ns}(K) = 1 + \sum_{j=1}^{\infty} a_j K^{j-1} \frac{d^j \theta_{ns}}{dK^j} \qquad (10)$$

where

$$a_j = -\frac{\Gamma(\beta)\Gamma(\frac{2-m}{\gamma}+1)}{C(m-1)} \times \sum_{i=j-1}^{\infty} \sum_{r=0}^{j} \binom{-\beta}{i} \binom{j}{r} \frac{(-1)^r \prod_{p=-i}^{0} \left(\frac{j-r}{\gamma}(m-1) + p\right)}{\Gamma(\frac{2-m}{\gamma} + \beta + i + 1)\Gamma(j+1)} \qquad (11)$$

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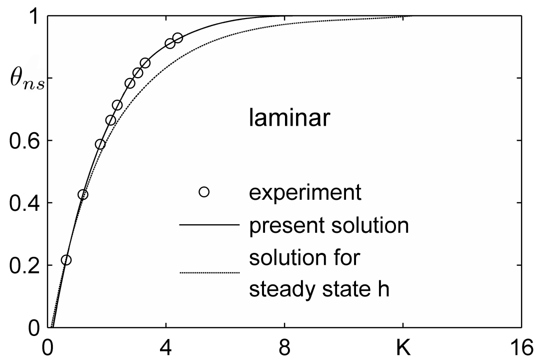

Fig. 4 Thin plate transient temperature and comparison to measurements. Laminar flow.

Eq. (10) is still an ordinary differential equation of infinite order. Its solution can be expressed in the form of a power series as

$$\theta_{ns}(K) = \sum_{i=1}^{\infty} c_i K^i \tag{12}$$

Substituting Eq. (12) into Eq. (10) and taking into account the initial condition $\theta_{ns}(0) = 0$ results in an iterative formula for c_i

$$c_1 = -a_1^{-1}$$

$$c_{i+1} = \left(\sum_{j=1}^{i+1} \frac{\Gamma(i+2)}{\Gamma(i+2-j)} a_j \right)^{-1} c_i, \quad i = 1, 2, 3, ... \tag{13}$$

In Fig. 4 the results from the present calculation for the transient heat transfer are compared to the measurements (Karvinen, 1978b), when the boundary layer is laminar. The numerical solution of Eq. (2) also gives the same result. In Fig. 4 the result (lower curve) is shown if use is made of a steady state heat transfer coefficient h for a uniform heat flux distribution from $Nu = hx/k = 0.46 Re_x^{1/2} Pr^{1/3}$. It can be seen that for a laminar boundary layer the effect of varying surface temperature is significant. For a turbulent boundary layer, the effect is very small. The measured results are obtained using a step change in heat generation in very thin aluminum ($b = 0.019mm$) and stainless steel ($b = 0.5mm$) sheets. The series solution was found to be well convergent at least up to the value $K = 10$. For values $K > 10$ there is no need for computation since θ_{ns} practically equals 1. The present results prove that the quasi-static approach is valid in practical applications where walls are much thicker than in measurements above in Fig. 4.

4. CONJUGATE HEAT TRANSFER WITH CONSTANT TEMPERATURE ON ONE SURFACE

The classical steady state conjugated problem in Fig. 2 can be treated in a very similar way to that described for a thin plate above. Eq. (1) still holds for the upper surface of the plate. For a relatively thin plate, the temperature distribution in the y-direction can be assumed to be linear (Luikov, 1974). Thus, the heat flux density from the upper surface of the plate can be assumed to be

$$q(x) = \frac{k_s}{b} (T(x) - T_b) \tag{14}$$

In the case of a short and thick plate the leading edge region should be treated as a two-dimensional problem where stagnation point convection is also included. The solution is sought in terms of the dimensionless variable

$$\theta_T(\tau) = \frac{T_b - T(x)}{T_b - T_\infty} \tag{15}$$

An asymptotic series solution for the θ_T has been found previously by Payvar (1977). The solution can be found by expanding θ_T as power series in the reciprocal of Br_x. The solution is

$$\theta_T(Br_x) = \sum_{i=0}^{\infty} d_i Br_x^{-i} \tag{16}$$

where $d_0 = 1$ and

$$d_{i+1} = -d_i \frac{B\left(\beta, \frac{1+i(1-m)}{\gamma}\right)}{C\gamma} \tag{17}$$

However, the applicability of Eq. (16) is limited because the series diverges for small Br_x. The solution presented below overcomes this limitation. Substituting Eq. (14) into Eq. (1) and performing the change of variables

$$\tau = Br_x^{\frac{\gamma}{m-1}} \tag{18}$$

and

$$\frac{\tau'}{\tau} = \left(\frac{s}{x}\right)^{\gamma} \tag{19}$$

one obtains

$$\theta_T(\tau) = 1 - \frac{\Gamma(\beta)\tau^{1-\beta-\frac{m}{\gamma}}}{C\gamma} D_\tau^{-\beta}\left[\tau^{\frac{1}{\gamma}-1}\theta_T\right] \tag{20}$$

Manipulating Eq. (20) as in Section 3, the following differential equation results

$$\theta_T(Br_x) = 1 + \sum_{j=0}^{\infty} a_j Br_x^{j-1} \frac{d^j\theta_T}{d(Br_x)^j} \tag{21}$$

where

$$a_j = -\frac{\Gamma(\beta)\Gamma(\frac{1}{\gamma})}{C\gamma} \times \sum_{i=j}^{\infty}\sum_{r=0}^{j} \binom{-\beta}{i}\binom{j}{r} \frac{(-1)^r \prod_{p=-i+1}^{0}\left(\frac{j-r}{\gamma}(m-1)+p\right)}{\Gamma(\frac{1}{\gamma}+\beta+i)\Gamma(j+1)} \tag{22}$$

The solution can again be expressed in the power series form as

$$\theta_T = \sum_{i=1}^{\infty} c_i Br_x^i \tag{23}$$

Substituting Eq. (23) into Eq. (21), the following iterative formula for c_i can be obtained

$$c_1 = \frac{-1}{a_0 + a_1}$$

$$c_{i+1} = \left(\sum_{j=1}^{i+1} \frac{\Gamma(i+2)}{\Gamma(i+2-j)} a_j \right)^{-1} c_i, \quad i = 1, 2, 3, ... \tag{24}$$

The series solution in Eq. (23) can be used for small Br_x but it diverges for large Br_x.

Results of the conjugated problem with a constant temperature at the lower surface are compared to those of other authors in Fig. 5. The present results were computed from Eqs. (22)-(24) for Br_x values below 1. It can be seen that the transition to large Br_x results of Payvar computed from Eqs. (16)-(17), happens smoothly. In the figure the small Br_x asymptotes of Luikov (1974) and Payvar (1977) are also shown.

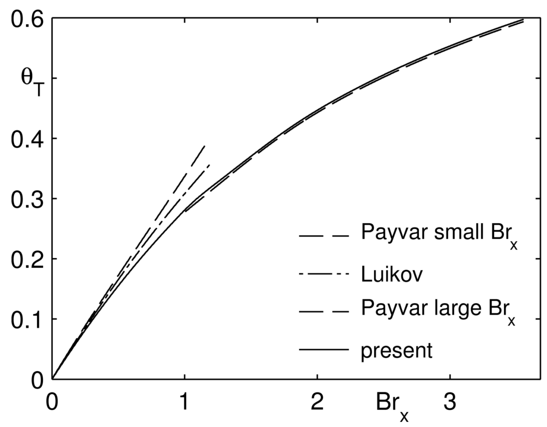

Fig. 5 Comparison of present solution with others (Luikov problem). The Payvar solution shown for $Br_x > 1$.

5. ELECTROLUMINESCENCE DISPLAY

The problem in Fig. 3 can also be solved using similar techniques to the two cases above. Assuming a linear temperature profile in the y-direction and using Eq. (1), the temperature distribution of the upper surface of the plate can be written as

$$T_2(x) - T_\infty = \frac{Re_x^{-m} Pr^{-n}}{Ck_f} \times \int_0^x \left[1 - \left(\frac{s}{x}\right)^\gamma\right]^{\beta-1} \frac{k_s}{b} \left[T_1(s) - T_2(s)\right] ds \quad (25)$$

On the other hand, for the lower surface temperature one obtains

$$T_1(x) - T_\infty = A \frac{Re_x^{-m} Pr^{-n}}{Ck_f} \times \int_0^x \left[1 - \left(\frac{s}{x}\right)^\gamma\right]^{\beta-1} \left(\phi'' - \frac{k_s}{b}\left[T_1(s) - T_2(s)\right]\right) ds \quad (26)$$

where

$$A = \frac{\left(\frac{Re_x^{-m} Pr^{-n}}{Ck_f}\right)_1}{\left(\frac{Re_x^{-m} Pr^{-n}}{Ck_f}\right)_2} \quad (27)$$

The parameter A obtains the value $A = 1$ in the usual case of identical flows on both surfaces.

It is relatively straightforward to obtain an asymptotic series solution for large Br_x. The solution is

$$\theta_q(Br_x) = \sum_{i=0}^{\infty} d_i Br_x^{-i} \quad (28)$$

where $d_0 = 1$ and

$$d_i = -d_{i-1} \frac{(A+1)}{C} B\left(\beta, \frac{1 + i(1-m)}{\gamma}\right) \quad (29)$$

The analysis to obtain a solution suitable for small values of Br_x is again slightly more involved. After substracting Eq. (25) from Eq. (26) and performing similar operations as in the previous section, one obtains the ordinary differential equation

$$\theta_q(Br_x) = A - (1 + A) \sum_{j=0}^{\infty} a_j Br_x^{j-1} \frac{d^j \theta_q}{d(Br_x)^j} \quad (30)$$

where

$$\theta_q(Br_x) = \frac{Ck_f \gamma Re_x^m Pr^n \left(T_1(x) - T_2(x)\right)}{B(\beta, \frac{1}{\gamma}) x \phi''} \quad (31)$$

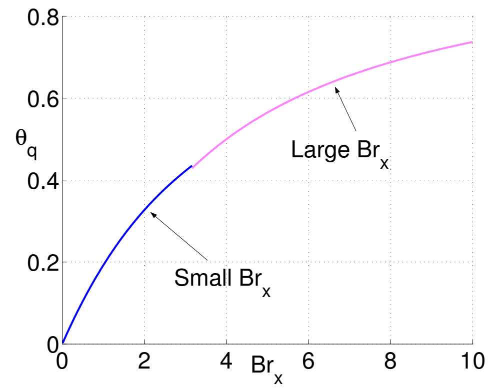

Fig. 6 Results for electroluminescence display ($A = 1$).

is the dimensionless temperature difference across the plate, which is defined so that $\theta_q = 1$ in the limit as $Br_x \to \infty$ and all the heat leaves the plate from the lower surface. The constants a_j in Eq. (31) are given by

$$a_j = \frac{\Gamma(\beta)\Gamma(\frac{2-m}{\gamma})}{C\gamma} \times \sum_{i=j}^{\infty} \sum_{r=0}^{j} \binom{-\beta}{i} \binom{j}{r} \frac{(-1)^r \prod_{p=-i+1}^{0} \left(\frac{j-r}{\gamma}(m-1) + p\right)}{\Gamma(\frac{2-m}{\gamma} + \beta + i)\Gamma(j+1)} \quad (32)$$

The solution of Eq. (30) can be found in the form

$$\theta_q(Br_x) = \sum_{i=1}^{\infty} c_i Br_x^i \quad (33)$$

in which the coefficients c_i are found from the following formula

$$c_1 = \frac{A}{(1+A)(a_0 + a_1)}$$

$$c_{i+1} = -\left((1+A) \sum_{j=0}^{i+1} \frac{\Gamma(i+2)}{\Gamma(i+2-j)} a_j\right)^{-1} c_i\,, \qquad i = 1, 2, 3, \ldots \quad (34)$$

The solution in Eq. (34) is again only suitable for small values of Br_x. The results for the last conjugated problem, uniform heat generation at lower surface and convective cooling on both surfaces of the plate (Fig. 3), are shown in Fig. 6. The results shown are for the parameter value $A = 1$, which corresponds to the most common case of identical boundary layers on both sides of the plate. The transition from the small Br_x asymptote to the large Br_x asymptote happens again smoothly. The small Br_x solution in Eqs. (32)–(34) is used for $Br_x < 3.2$ and the large Br_x solution in Eqs. (28)–(29) is used for $Br_x > 3.2$.

6. CONCLUSIONS

Analytical solutions for a class of conjugated convective heat transfer problems have been developed. Some of the results have also been compared with those existing in the literature.The solutions are based on the superposition method in which an analytical exspression is used to couple together an arbitrarily varing surface heat flux and surface temperature in a stream-wise direction. This type of approach leads to integral equations which have been solved with the help of the Riemann- Liouville integral operator.

Three specific examples are solved: transient temperature of a thin plate with a step heat input, the traditional Luikov problem and the modified plate problem with a uniform heat generation on one plate surface and convectively cooled at both surfaces.The last problem is a practical application of an electroluminescence display production process.

As to the importance and generality of the results following can be mentioned: The transient solution of a thin plate with a uniform step heat flux shows that these types of problem can be solved using a quasi-steady approach in which convection is treated as a steady-state but the stream-wise variation of the surface temperature is taken into account. The solution presented is correct for every practical application, because the solution is given using a non-dimensional variable which includes all thermal,physical and geometrical properties.The same is also true for the solutions of the Luikov problem and the modified problem.

If the boundary layer is turbulent a fixed heat transfer coefficient based on an isothermal surface can be used but for a laminar boundary layer, the effect of surface temperature distribution must be taken into account.

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NOMENCLATURE

A	ratio of boundary layer properties, given by Eq. (26)
a_j	coefficients in differential equations
b	plate thickness
B	Beta function
Br_x	Brun number, $\frac{k_f}{k_s}\frac{b}{x}Re_x^m Pr^n$
C	constant in Eq. (1)
c	coefficients in power series solutions
c_p	heat capacity of the plate
D	Riemann-Liouville fractional integration operator
d	coefficients in power series solutions
i, j	summation indices
k	thermal conductivity
K	dimensionless variable, $\frac{k_f Re_x^m Pr^n t}{x\rho c_p b}$
m, n	parameters in Eq. (1)
Pr	Prandtl number
p	product index
q	convective heat flux
r	summation index
Re_x	Reynolds number, $U_\infty x/\nu$
s	dummy variable in integral Eq. (1)
T	temperature
t	time
U_∞	free steam velocity
x	streamwise coordinate
y	coordinate normal to plate
z	general variable
Greek Symbols	
β, γ	constants in Eq. (1)
ϕ''	heat generation per surface area
ϕ'''	heat generation per volume
Γ	Gamma function
λ	general exponent
ρ	density of the plate
θ_{ns}	dimensionless transient temperature defined by Eq. (3)
θ_q	dimensionless temperature difference across the plate defined by Eq. (31)
θ_T	dimensionless temperature difference across the plate defined by Eq. (15)
τ	variable used to simplify solution, defined by Eqs. (5) and (18)
τ'	dummy variable defined by Eqs. (6) and (19)
Subscripts	
1	lower surface of plate
2	upper surface of plate
f	fluid
s	solid
ss	steady state

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APPENDIX: FRACTIONAL CALCULUS AND DIFFERENTIATION OF COMPOSITE FUNCTIONS

Certain type of integrals can be presented with the help of the Riemann-Liouville integral operator D, which is defined as (Osler, 1970b; Tu *et al.*, 2001):

$$D_z^{-\beta} f(z) = \frac{1}{\Gamma(\beta)} \int_0^z \frac{f(\xi) d\xi}{(z-\xi)^{1-\beta}} \tag{35}$$

The definition in Eq. 35 is valid for every positive real number β. The definition can easily be extended for negative or even complex values of β but it is not necessary for purposes of this paper. Fractional calculus is a generalisation of ordinary differential calculus. For positive integer values of β the definition in Eq. 35 reduces to βth-order integration of the function $f(z)$. This fact can easily be proven by repeated partial integration.

The Riemann-Liouville integral of a power function is given by

$$D_z^{-\beta} x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\beta+1)} z^{\lambda+\beta} \tag{36}$$

The Riemann-Liouville operators have several properties that make it easier to handle the integral equations. One of the most important of them is the generalised Leibniz rule, (Osler, 1970b; Tu *et al.*, 2001):

$$D_z^{-\beta} \left(f(z) g(z) \right) = \sum_{i=0}^{\infty} \binom{-\beta}{i} D_z^{-\beta-j} \left\{ g(z) \right\} \frac{d^i}{dz^i} \left\{ f(z) \right\} \tag{37}$$

where

$$\binom{-\beta}{i} = \frac{\Gamma(1-\beta)}{\Gamma(1-\beta-i)\Gamma(1+i)} \tag{38}$$

is the generalised binomial coefficient.

In Eq. (37) there exists ith order differentiation of the function $f(z)$, where i is a positive integer. It frequently occurs that the function $f(z)$ is a composite function, that is $f(z) = F(h(z))$. In this case the differentiation can be carried out with the help of the following chain rule formula (Osler, 1970a):

$$\frac{d^i}{dz^i} f(z) = \sum_{j=0}^{i} \frac{U_j(z)}{j!} \frac{d^j}{d(h(z))^j} f(z) \tag{39}$$

where

$$U_j(z) = \sum_{r=0}^{j} \binom{j}{r} \left(-h(z) \right)^r \frac{d^i}{dz^i} \left(h(z) \right)^{j-r} \tag{40}$$